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INSTABILITIES IN URANIUM PLASMA
AND THE
GAS-CORE NUCLEAR ROCKET ENGINE

D. A. Tidman
VERSAR INC.
6621 Electronic Drive
Springfield, Virginia 22150

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ABSTRACT

The nonlinear evolution of unstable sound waves in a uranium plasma has been calculated using a multiple time-scale asymptotic expansion scheme. The fluid equations used include the fission power density, radiation diffusion, and the effects of the changing degree of ionization of the uranium atoms. The nonlinear growth of unstable waves is shown to be limited by mode coupling to shorter wavelength waves which are damped by radiation diffusion. This mechanism limits the wave pressure fluctuations to values of order $\delta P/P \sim 10^{-5}$ in the plasma of a typical gas-core nuclear rocket engine. The instability is thus not expected to present a control problem for this engine.

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REFERENCES

1. INTRODUCTION

The objective of this work is to analyze the nature of instabilities in fission reactors in which the uranium operates at a temperature sufficiently high so as to become ionized. The principal impetus for study of such uranium plasma reactors has derived from the concept of a high specific impulse nuclear gas-cored rocket engine.^(1,2) However, fissioning plasma reactors may also lead to development of a new class of magnetohydrodynamic generators⁽³⁾, and perhaps may even have application to the direct nuclear excitation of lasers.

The equations describing a uranium plasma involve the coupling of the plasma "fluid" dynamics to the neutron transport and also to the radiation transport in the system. In such a reactor, the plasma would be enclosed (or nearly enclosed) by a neutron moderator — which is important in that it returns the thermalized neutrons back to the plasma core — while radiation transport dominates the transport of energy within the plasma. As a theoretical structure, then, the coupled system of equations can be charted symbolically as:

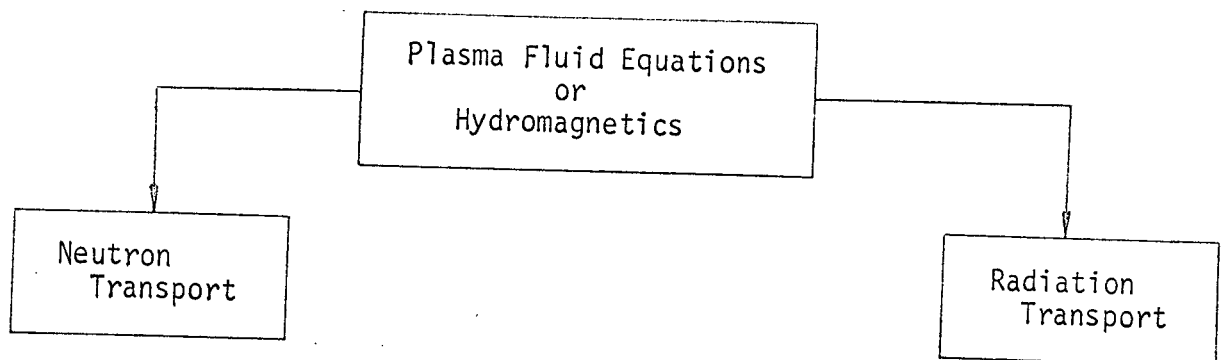


Figure 1.1

The specific set of equations involved are listed as equations (2.24)-(2.28) in the next section of this report.

The major application for the fissioning uranium plasma reactor (on which significant effort has been expended) is that of the gas-cored nuclear rocket engine shown below.

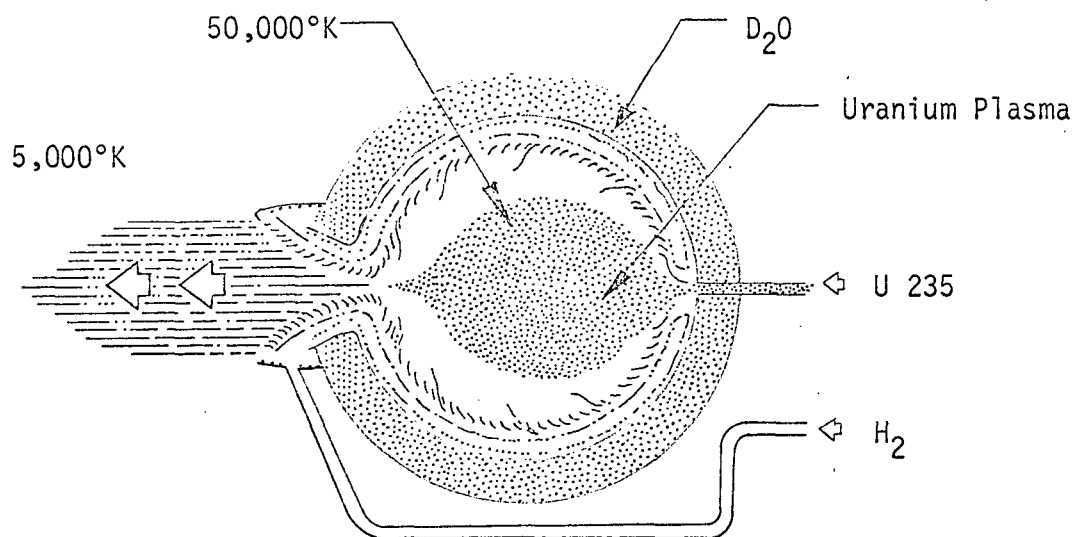


Figure 1.2

In general, this system consists of a chamber in which a fissioning uranium plasma (at $\sim 5 \cdot 10^4$ °K) is maintained in the central region and radiatively heats the propellant hydrogen gas which flows around the main core and is subsequently ejected from the rocket nozzle. The chamber is surrounded by a moderator which slows down the fission neutrons and returns them to the uranium core.

It is clear that if instabilities occur in this device, leading to either a fine-scale plasma turbulence or larger scale oscillations, they could well have undesirable results — such as enhancing the uranium fuel ejection from the rocket. An understanding of these processes (and their possible controllability) is of basic importance for the feasibility of constructing

such plasma-cored rocket engines.

In a more general context, it should also be noted that the direct excitation of plasma motion and plasma waves by a fissioning plasma reactor may have useful applications to other energy conversion devices such as MHD generators.

One source of instability is the growth of sound waves^(4,5) in the plasma due to the fission power density, P_{fiss} . The instability mechanism involved can be understood as follows: Consider a standing sound wave in a bounded region of fissioning plasma with a constant background thermal neutron density. In the wave compressions the fission power density, P_{fiss} , increases due to the increased uranium density, and in the rarefactions P_{fiss} decreases. This results in an increased pressure gradient associated with the wave which in turn leads to a transfer of fission power to the wave. It occurs because the wave compression tends to expand more rapidly than it was compressed. However, competing with this is the fact that radiation tends to transport the extra thermal energy out of the wave compressions. Radiation diffusion smooths out the temperature fluctuations of waves more rapidly the shorter their wavelength. This results in a critical wavelength, λ_{crit} , below which waves are stable and above which they are unstable.

We have calculated the nonlinear evolution of such unstable waves to determine their limiting amplitude. The mechanism that finally limits their amplitude is the mode coupling of unstable waves in the wavelength range $\lambda > \lambda_{crit}$ to radiation damped waves in the range $\lambda < \lambda_{crit}$.

Our conclusion is that this mechanism "turns off" the instability after the wave amplitudes have reached relatively small values for most systems of interest. Relative pressure fluctuations of order $\delta P/P \sim 10^{-7}$ to 10^{-5} are calculated to occur for a slab of uranium plasma with parameters corresponding to the NASA-Lewis design engine. Consequently this instability is not expected to present a control problem for this engine. The instability would only become a problem for devices of a size much larger than those currently contemplated.

A second mechanism for instability derives from the fact that the fission power density in the reactor cavity depends on the relative fluid velocity between the neutron gas and the uranium plasma. For low neutron thermal energies (below 1 eV which is the lowest neutron energy for resonant capture by U_{235}) the neutron induced fission power increases as the relative fluid velocity increases. This "Doppler effect" enhances the fission power in some parts of a sound wave and not others and can give rise to growth of the wave. This process is analysed in Chapter 7. Our conclusion is that nonlinear effects also limit the amplitude of waves in this case to small values which are not expected to cause problems for the gas-core engine.

2. DERIVATION OF FLUID EQUATIONS FOR FISSIONING U-PLASMA

In this section the fluid equations for a uranium plasma will be derived from the equations for the particle distribution functions. This has the advantage that one can see physically the assumptions that are involved in the fluid description and where future generalizations might be made.

Define a distribution function F so that

$$F(\alpha, \underline{v}, \underline{x}, t) d\underline{v} d\alpha = (\text{average number of U atoms or ions in } d\underline{v} \text{ in states } \alpha, d\alpha.) \quad (2.1)$$

If the states labelled α are discrete for part of the α range, then F involves a series of δ -functions in α . The corresponding velocity distribution function for U atoms in any charge state is

$$f = \int F d\alpha, \quad (2.2)$$

and the total number density for all U atoms and ions is

$$N_u = \int f d\underline{v} \equiv \sum_i N_u^{(i)}, \quad (2.3)$$

where i goes over all the partial densities for ions that have lost i electrons.

Similarly the fluid velocity \underline{V}_u and pressure tensor $P_{u\underline{\underline{I}}}$ for the U-component of the plasma are written

$$N_u \underline{V}_u = \int d\underline{v} \underline{v} f, \quad (2.4)$$

$$P_{u\underline{\underline{I}}} = N_u K T \underline{\underline{I}} = M \int d\underline{v} f (\underline{v} - \underline{V}_u) (\underline{v} - \underline{V}_u), \quad (2.5)$$

where we have assumed a local velocity distribution close to a Maxwellian

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with a temperature T . Viscosity, which derives from the off-diagonal parts of \underline{P}_u , will be neglected.

Now the transport equation is

$$\frac{\partial F}{\partial t} + \underline{v} \cdot \frac{\partial F}{\partial \underline{x}} = C_u \quad (2.6)$$

where $C_u \, d\underline{x} \, d\underline{v} \, d\alpha$ = rate of change of U-atoms in $d\underline{x} \, d\underline{v} \, d\alpha$ due to atomic processes, fission processes, and interaction with the radiation field (*i.e.*, C includes all microscopic effects).

Taking moments $\int d\underline{v} \, d\alpha$ and $\int \underline{v} \, d\underline{v} \, d\alpha$ of (2.6) gives the mass and momentum transport equations,

$$\frac{\partial N_u}{\partial t} + \frac{\partial}{\partial \underline{x}} \cdot (N_u \underline{v}_u) = \int C_u \, d\underline{v} \, d\alpha, \quad (2.7)$$

$$\frac{\partial}{\partial t} (N_u \underline{v}_u) + \frac{1}{M} \frac{\partial}{\partial \underline{x}} (N_u K T) + \frac{\partial}{\partial \underline{x}} \cdot (N_u \underline{v}_u \underline{v}_u) = \int C_u \, \underline{v} \, d\underline{v} \, d\alpha. \quad (2.8)$$

For the energy transport equation we operate on (2.6) with

$$\underbrace{\int d\underline{v} \, d\alpha \left[\frac{1}{2} M (\underline{v} - \underline{v}_u)^2 + \varepsilon(\alpha) \right]}_{\left(\begin{array}{l} \text{thermal energy} \\ \text{density excluding} \\ \text{translational K.E.} \end{array} \right)} \quad \underbrace{\dots}_{\left(\begin{array}{l} \text{"internal energy" of} \\ \text{ionization excitation} \\ \text{and radiation} \end{array} \right)}$$

Writing

$$\int d\underline{v} \, d\alpha \frac{\partial F}{\partial t} \varepsilon(\alpha) = \frac{\partial}{\partial t} (N \bar{\varepsilon}), \quad (2.9)$$

$$q_u = \frac{M}{2} \int d\underline{v} (\underline{v} - \underline{v}_u) (\underline{v} - \underline{v}_u)^2 f = \text{heat flux} ,$$

the resulting energy transport equation becomes,

$$\begin{aligned} \frac{d}{dt} \left(\frac{3}{2} N_u K T \right) + \frac{\partial}{\partial t} (N_u \bar{\epsilon}) + \frac{\partial}{\partial \underline{x}} \cdot (N_u \underline{v} \bar{\epsilon}) + \frac{5}{2} N_u K T \frac{\partial}{\partial \underline{x}} \cdot \underline{v}_u + \frac{\partial}{\partial \underline{x}} \cdot \underline{q}_u \\ = \int d\underline{v} d\alpha C_u \left[\frac{1}{2} M (\underline{v} - \underline{v}_u)^2 + \epsilon \right] , \end{aligned} \quad (2.10)$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{v}_u \cdot \frac{\partial}{\partial \underline{x}} \quad (2.11)$$

In order to obtain a closed set of 1-fluid equations we must next consider the meaning of the collision terms in (2.7) - (2.10) and the role of the electrons.

Number Density Equation

There is no local source of U atoms, and we shall neglect the local sink due to fission. We also neglect the accumulation of a spectrum of fission fragments which would form a new ion species. Under these circumstances $\int C_u d\underline{v} d\alpha \approx 0$, so that equation (2.7) gives

$$\frac{\partial N_u}{\partial t} + \frac{\partial}{\partial \underline{x}} \cdot (N_u \underline{v}_u) = 0 \quad (2.12)$$

for the continuity equation for U number density.

However, for the electrons there are sources and sinks due to the rapid atomic processes of recombination and ionization. These atomic processes

occur on time scales much less than those associated with the fluid dynamics. Thus a value for the electron number density N_e will be assumed corresponding to local thermodynamic equilibrium

$$N_e = N_u \bar{Z} \quad (2.13)$$

where \bar{Z} is the mean charge state of the U atoms and must be calculated from the Saha formulas.

Momentum Equation

Fission does not change the average local momentum density of the plasma. The electrons and ions however couple strongly through collisions so that $\underline{V}_e = \underline{V}_u$, $T_e = T_u = T$, and $\sum_{e,i} m \int \underline{v} d\underline{v} C = 0$.

Thus adding (2.8) to it's counterpart for the electrons and treating $m/M_u \ll 1$ gives

$$M \frac{\partial}{\partial t} (N_u \underline{V}_u) + \frac{\partial}{\partial \underline{x}} \cdot [(N_u + N_e)KT] + M \frac{\partial}{\partial \underline{x}} \cdot (N_u \underline{V}_u \underline{V}_u) = 0 \quad (2.14)$$

Note, we could include a photon pressure in (2.14) but have not done so at this point.

Energy Equation

Now the local energy density (excluding only translational kinetic energy) is

$$\int d\underline{v} d\alpha \left[(\epsilon_{\text{exct.}} + \epsilon_{\text{ioniz}} + \epsilon_{\text{Rad}}) + \frac{1}{2} M (\underline{v} - \underline{V}_u)^2 \right] f, \quad (2.15)$$

where the quantity ϵ in (2.10) has been split up into its contributions from

internal energy of excitation, ionization and radiation. The ionization energy is viewed as similar to internal excitation energy for the atoms. The various atomic processes involved undergo energy interchanges with the thermal velocity component and not the ordered kinetic energy density of the plasma, which is why we chose to take the above moment for the energy equation. The quantity (2.15) changes due to collisions, but when summed over atoms and electrons its remaining changes (due to C) are those due to fission and radiation, *i.e.*,

$$\begin{aligned} \sum_{e,u} \int d\underline{v} \, d\alpha \, C \left[(\epsilon_{\text{exct}} + \epsilon_{\text{ioniz}} + \epsilon_{\text{Rad}}) + \frac{1}{2} M(\underline{v} - \underline{v}_u)^2 \right] f \\ = P_{\text{fiss}} + P_v \end{aligned} \quad (2.16)$$

where

$$\begin{cases} P_{\text{fiss}} &= \text{fission power density in frame of the U-plasma.} \\ P_v &= \text{nett local radiation energy absorption less emission} \end{cases} \quad (2.17)$$

Adding the electron version of (2.10) to the ion energy equation (2.10) then gives the total energy transport equation for the fluid,

$$\begin{aligned} \frac{d}{dt} \left[\frac{3}{2} N_u K T (1 + \bar{Z}) \right] + \frac{\partial}{\partial t} (N_u \bar{\epsilon}) + \frac{\partial}{\partial \underline{x}} \cdot (\underline{v} N_u \bar{\epsilon}) + \frac{5}{2} (1 + \bar{Z}) N_u K T \frac{\partial}{\partial \underline{x}} \cdot \underline{v}_u \\ + \frac{\partial}{\partial \underline{x}} \cdot (\underline{q}_u + \underline{q}_e) = P_{\text{fiss}} + P_v \end{aligned} \quad (2.18)$$

The thermal conduction terms \underline{q}_u and \underline{q}_e will be neglected since the most important energy transport is due to radiation as expressed by P_v . The

quantities P_{fiss} and P_v are next required in order to complete the fluid equations.

Fission Power Density, P_{fiss}

Suppose v_0 is the average thermal velocity of the neutrons, N_0 their number density, and σ the cross-section for neutron induced fission events in which an energy Q is released in fission fragments. The fission power density then follows from

$$P_{\text{fiss}} \, d\underline{x} \, dt = \frac{(N_0 v_0 \, d\underline{x} \, dt \, Q)}{(1/N_u \sigma)} = \frac{\left(\begin{array}{c} \text{total path traveled} \\ \text{by } n_0 \text{'s in } d\underline{x} \, dt \end{array} \right) Q}{(\text{m.f.p. for fission})} ,$$

i.e.,

$$P_{\text{fiss}} = N_0 N_u v_0 \sigma Q \quad . \quad (2.19)$$

We have assumed σ to be independent of the relative velocity between the neutrons and U nuclei and taken a value for it averaged over the neutron distribution function, *i.e.*, $\sigma = \sigma(v_0, T)$. A more precise form for P_{fiss} would be

$$P_{\text{fiss}} = \iint d\underline{v}_0 \, d\underline{v}_u \, |\underline{v}_0 - \underline{v}_u| Q' f_0(\underline{v}_0) f_u(\underline{v}_u) \sigma |\underline{v}_u - \underline{v}_0| \quad (2.20)$$

where Q' is the fission kinetic energy release in the rest frame of the plasma. f_0 and f_u are velocity distribution functions for the neutrons and U nuclei. The simpler model (2.19) will be used for most purposes.

Radiation Transport term, P_v

In the interior region of the U plasma the radiation absorption length is usually much less than the gradient scales of the plasma, *i.e.*, the medium is optically thick. Under these circumstances the absorption coefficient can be approximated by the Rosseland mean value, K_R , given by

$$K_R = \frac{16}{3} \frac{\sigma T^3}{k_R} \quad , \quad (2.21)$$

where σ is the Stefan-Boltzmann constant, T the temperature, and k_R the Rosseland mean opacity. The radiation energy flux is then given by

$$\underline{S} = - K_R \underline{\nabla} T \quad (2.22)$$

and the divergence of this, $-\underline{\nabla} \cdot \underline{S}$, gives the local radiation power, P_v , *i.e.*,

$$P_v = \underline{\nabla} \cdot [K_R \underline{\nabla} T] \quad . \quad (2.23)$$

The quantity K_R is thus an effective thermal conductivity due to radiation transport.

Summary of Fluid Equations

Equations (2.12), (2.13), (2.18), (2.19) and (2.23) now form a complete set of fluid equations. Since they are no longer necessary, we shall drop the subscripts u on the total ion number density N_u and fluid velocity \underline{V}_u , and also drop the bars over $\bar{\epsilon}$ and \bar{Z} . The resulting system is,

$$\frac{\partial N}{\partial t} + \underline{\nabla} \cdot (N \underline{V}) = 0 \quad (2.24)$$

$$\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \frac{\partial}{\partial \underline{x}} \underline{V} + \frac{1}{MN} \frac{\partial}{\partial \underline{x}} [NKT (1 + Z)] = 0 \quad (2.25)$$

$$\begin{aligned} \frac{d}{dt} \left[\frac{3}{2} NKT (1 + Z) \right] + \frac{\partial}{\partial t} (N\epsilon) + \frac{\partial}{\partial \underline{x}} \cdot (N\epsilon \underline{V}) + \frac{5}{2} (1 + Z) NKT \frac{\partial}{\partial \underline{x}} \cdot \underline{V} \\ = P_{fiss} + \underline{V} \cdot (K_R \underline{V} T) \end{aligned} \quad (2.26)$$

$$P_{fiss} = N N_0 v_0 \sigma Q \quad (2.27)$$

$$K_R = \frac{16}{3} \frac{\sigma T^3}{k_R} \quad (2.28)$$

where Z , ϵ , must be calculated from the Saha relations for a thermal equilibrium U plasma.

Comments on the Energy Equation and Specific Heats

If we write the total "internal energy" (which includes thermal kinetic energy) as

$$E = \frac{3}{2} NKT (1 + Z) + N\epsilon \quad (2.29)$$

then equation (2.26) can also be written

$$\frac{dE}{dt} + E \frac{\partial}{\partial \underline{x}} \cdot \underline{V} + P \frac{\partial}{\partial \underline{x}} \cdot \underline{V} = 0, \quad (2.30)$$

where the pressure $P = (1 + Z)NKT$. This is the usual form of the energy equation (see Chapman & Cowling, Mathematical Theory of Nonuniform Gases, p.52, Eq. 4).

One can also derive an equation that expresses local energy conservation for the total energy. Multiplying (2.25) by MNV , adding the result to (2.26) and using (2.24) leads to

$$\begin{aligned} \frac{\partial}{\partial t} \left\{ \frac{1}{2} MNV^2 + \frac{3}{2} NKT (1 + Z) + N\varepsilon \right\} \\ + \frac{\partial}{\partial x} \cdot \left\{ \underline{V} \frac{1}{2} MNV^2 + \frac{5}{2} \underline{V} NKT (1 + Z) + N\underline{V}\varepsilon - K_R \frac{\partial T}{\partial x} \right\} \\ = P_{fiss} \end{aligned} \quad (2.31)$$

This shows that the total energy density in the medium is

$$E_{tot.} = \frac{1}{2} MNV^2 + E \quad (2.32)$$

The specific heats can be expressed in terms of these quantities (see Landau and Lifshitz, Stat. Mech. p. 45, or Chapman and Cowling, p.39). Thus

$$C_V = \left(\frac{\partial E}{\partial T} \right)_V, \quad C_P = \left(\frac{\partial [E + Pv]}{\partial T} \right)_P \quad (2.33)$$

where v is the gas volume. From (2.29) we see

$$C_V = NK \left[\frac{3}{2} (1 + Z + TZ_T) + \frac{\varepsilon_T}{K} \right] \quad (2.34)$$

The effective number of degrees of freedom, n , in the plasma can be defined by

$$E \equiv \frac{1}{2} n NKT, \quad (2.35)$$

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so from (2.29)

$$n = 3 (1 + Z) + \frac{2\varepsilon}{KT} . \quad (2.36)$$

However, since n is a function of T , this is not a particularly useful quantity in that the quantity $\gamma \equiv C_p/C_v \approx (1 + \frac{2}{n})$.

The first term, dE/dt , of (2.30) can also be written $C_v dT/dt$.

3. NUMERICAL RESULTS FOR THE AVERAGE IONIZATION ENERGY ϵ AND AVERAGE CHARGE NUMBER Z OF THE U-IONS

The functions ϵ and Z in this section are derived from some computer results kindly supplied by Dr. R. T. Schneider⁽⁶⁾. They are based on the UPLAZ-2 program (developed by the group in the Department of Nuclear Engineering Sciences, University of Florida) which calculates the composition of uranium plasma as a function of plasma pressure and temperature assuming local thermodynamic equilibrium.

The average ionization energy ϵ is defined as

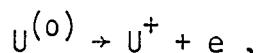
$$\epsilon = \frac{\sum N_u^{(i)} \epsilon_i}{\sum N_u^{(i)}} \quad (3.1)$$

where the sum goes over all ion charge states, and ϵ_i is the total energy required to produce an ion which has lost i electrons (including energy shifts due to the density effect), and $N_u^{(i)}$ is the number of ions in ionization state i per cm^3 .

The energy ϵ_i can be written as a sum due to successive ionizations,

$$\epsilon_i = (E_1 - \Delta E_1) + (E_2 - \Delta E_2) + \dots (E_i - \Delta E_i), \quad \epsilon_0 = 0, \quad (3.2)$$

where E_1 is the energy required for the process



and E_2 for $U^+ \rightarrow U^{++} + e$, etc., and the ΔE 's give the decrease in these energies due to the density effect.

The average charge number Z of the ions is also required and is defined by

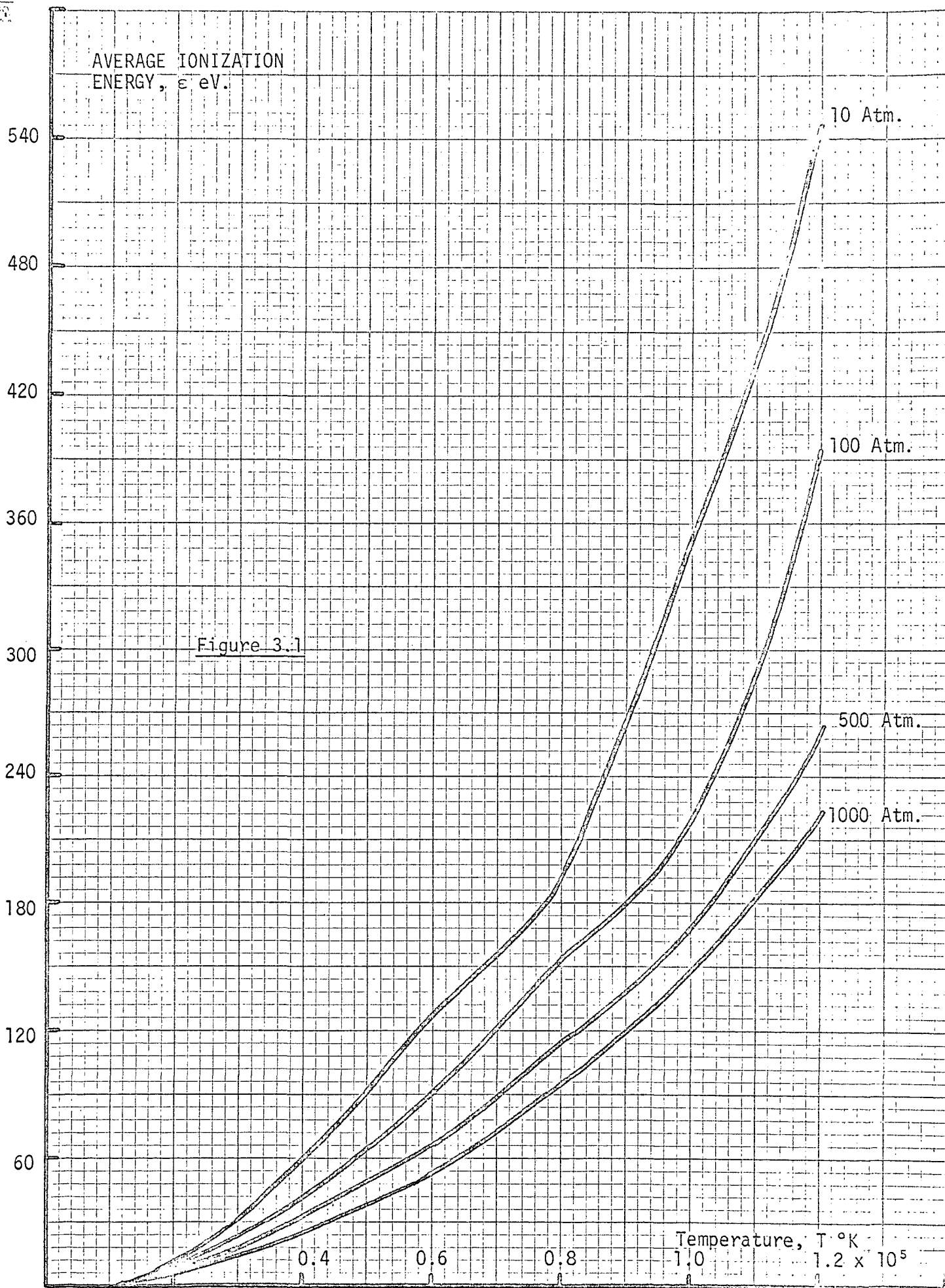
$$Z = \frac{\sum Z_i N_u^{(i)}}{\sum N_u^{(i)}} \quad (3.3)$$

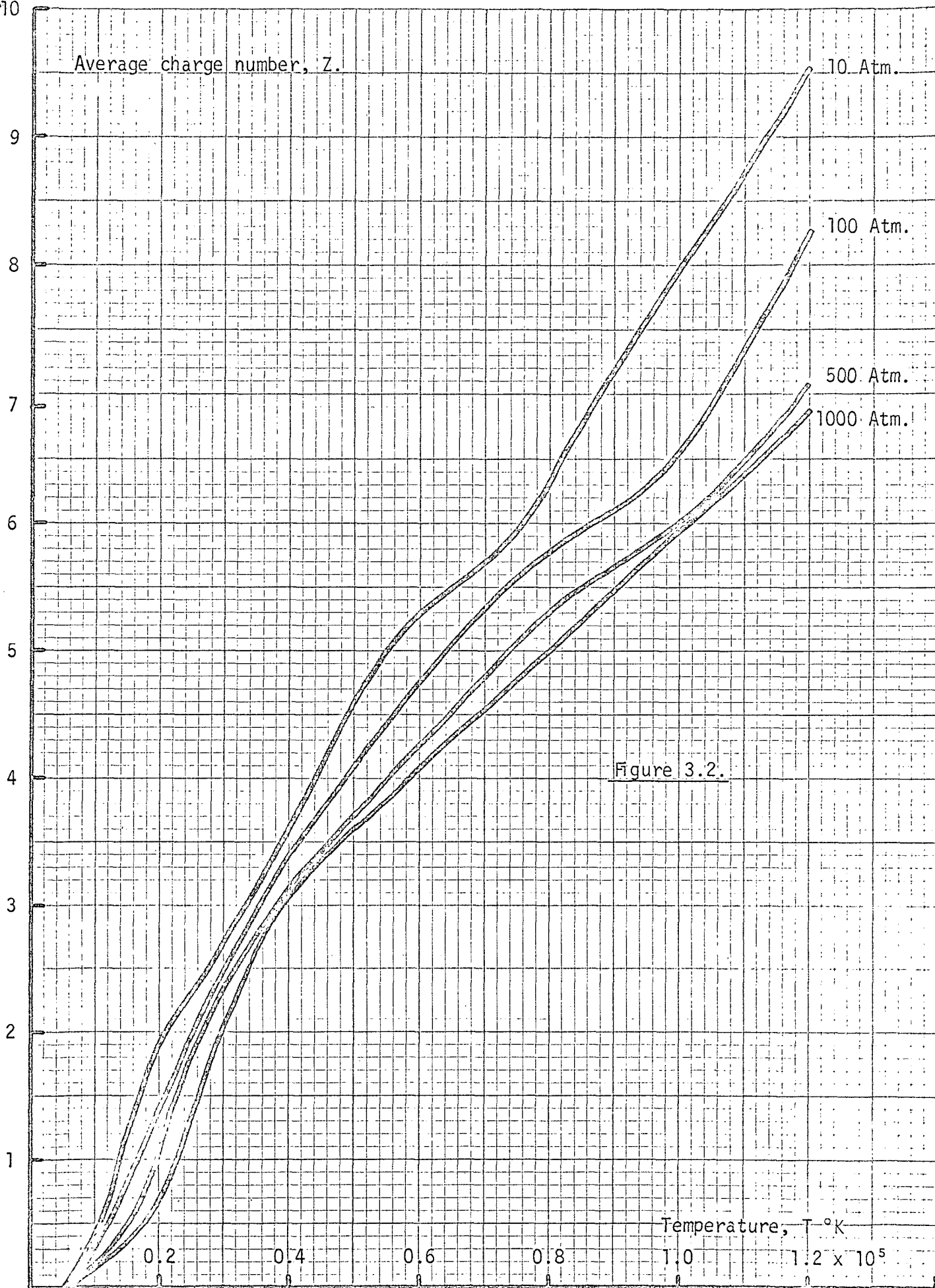
The quantities $N_u^{(i)}$, ΔE_i , $\sum Z_i N_u^{(i)}$ and $(Z + 1)N_u = N_{\text{total}}$, were supplied as functions of T and P_{tot} (total pressure) by UPLAZ-2, and after some computation the tables and graphs 3.1 and 3.2 of $\epsilon(T, P_{\text{tot}})$ and $Z(T, P_{\text{tot}})$ were obtained.

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T	$P_{\text{tot}} = 10 \text{ Atm.}$ Z	$P_{\text{tot}} = 100 \text{ Atm.}$ Z	$P_{\text{tot}} = 500 \text{ Atm.}$ Z	$P_{\text{tot}} = 1000 \text{ Atm.}$ Z
120,000	9.552	8.273	7.053	6.850
100,000	7.951	6.490	6.058	6.058
80,000	6.220	5.843	5.375	4.993
60,000	5.421	4.810	4.231	4.139
40,000	3.692	3.463	3.168	3.159
20,000	1.913	1.368	.848	.682
8,000	.159	.057	.031	.025
5,000	.006	.002	.001	.001

T	$P_{\text{tot}} = 10 \text{ Atm.}$ $\epsilon \text{ eV}$	$P_{\text{tot}} = 100 \text{ Atm.}$ ϵ	$P_{\text{tot}} = 500 \text{ Atm.}$ ϵ	$P_{\text{tot}} = 1000 \text{ Atm.}$ ϵ
120,000	552	383	261	221
100,000	363	210	167	154
80,000	188	153	117	93
60,000	129	89.4	63.5	55
40,000	55	42.9	31.2	28.2
20,000	16.6	9.1	4.6	3.5
8,000	.92	.31	.17	1.3
5,000	$4 \cdot 10^{-2}$	10^{-2}	$7 \cdot 10^{-3}$	$5 \cdot 10^{-3}$





4. DISPERSION RELATIONS FOR SOUND WAVES IN FISSIONING URANIUM PLASMA

The first step in our analysis is to derive the dispersion relations for sound waves in a fissioning U-plasma by linearizing the fluid equations in the wave amplitudes. The plasma in which the waves propagate is assumed to be stationary in time and approximately spatially homogeneous on the wavelength scale of the waves being considered.

The fluid equations are (dropping subscripts u and bars on ϵ and Z),

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x} \cdot (N \underline{V}) = 0 \quad (4.1)$$

$$\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \frac{\partial}{\partial x} \underline{V} + \frac{1}{MN} \frac{\partial}{\partial x} [NKT(1 + Z)] = 0 \quad (4.2)$$

$$\left(\frac{\partial}{\partial t} + \underline{V} \cdot \frac{\partial}{\partial x} \right) \left[\frac{3}{2} NKT(1 + Z) + N\epsilon \right] + \frac{5}{2} (1 + Z) NKT \frac{\partial}{\partial x} \cdot \underline{V} \quad (4.3)$$

$$+ N\epsilon \frac{\partial}{\partial x} \cdot \underline{V} = P_{fiss} + \underline{\nabla} \cdot (K_R \underline{\nabla} T)$$

where

$$\begin{cases} P_{fiss} = N N_0 v_0 \sigma Q \\ Z, \epsilon, K_R \text{ are given fns of } T, N \end{cases} \quad (4.4)$$

Now write the fluid variables as

$$\left\{ \begin{array}{l} N = N_0 + N_1 \\ \underline{V} = \underline{0} + \underline{V}_1 \\ T = T_0 + T_1 \\ Z = Z_0 + N_1 Z_N + T_1 Z_T \\ \epsilon = \epsilon_0 + N_1 \epsilon_N + T_1 \epsilon_T \end{array} \right. , \quad (4.5)$$

where $Z_N = (\partial Z / \partial N)_{N=N_0, T=T_0}$ etc., and the equilibrium state is denoted by subscripts 0 and perturbations by subscripts 1.

The linearized equations then follow as,

$$\frac{\partial N_1}{\partial t} + N_0 \frac{\partial}{\partial \underline{x}} \cdot \underline{V}_1 = 0 \quad (4.6)$$

$$\frac{\partial \underline{V}_1}{\partial t} + \frac{1}{MN_0} \frac{\partial}{\partial \underline{x}} \cdot \left[N_1 K T_0 (1 + Z_0 + N_0 Z_N) + N_0 K T_1 (1 + Z_0 + T_0 Z_T) \right] = 0 \quad (4.7)$$

$$\begin{aligned} & \frac{3}{2} \frac{\partial}{\partial t} \left[N_1 K T_0 (1 + Z_0 + N_0 Z_N) + N_0 K T_1 (1 + Z_0 + T_0 Z_T) \right] \\ & + \frac{\partial}{\partial t} \left[N_1 \epsilon_0 + N_0 N_1 \epsilon_N + N_0 T_1 \epsilon_T \right] + N_0 \epsilon_0 \frac{\partial}{\partial \underline{x}} \cdot \underline{V}_1 \\ & + \frac{5}{2} (1 + Z_0) N_0 K T_0 \frac{\partial}{\partial \underline{x}} \cdot \underline{V}_1 = \frac{N_1}{N_0} P_{fiss,0} + K_{R0} \nabla^2 T_1 \end{aligned} \quad (4.8)$$

Next, consider Fourier modes

$$(N_1, \underline{V}_1, T_1) \rightarrow (N_1, \underline{V}_1, T_1) e^{i(\underline{k} \cdot \underline{x} - \omega t)}$$

Thus

$$- \omega N_1 + N_0 \underline{k} \cdot \underline{V}_1 = 0 \quad (4.9)$$

$$- \omega \underline{V}_1 + \frac{1}{MN_0} \underline{k} (AN_1 + BT_1) = 0 \quad (4.10)$$

$$\left\{ \begin{aligned} & N_1 \left\{ - \omega \left(\frac{3}{2} A + \epsilon_0 + N_0 \epsilon_N \right) + \frac{iP_{fiss,0}}{N_0} \right\} \\ & + T_1 \left\{ - \omega \left(\frac{3}{2} B + N_0 \epsilon_T \right) - ik^2 K_{R0} \right\} \\ & + \left[\frac{5}{2} (1 + Z_0) N_0 K T_0 + N_0 \epsilon_0 \right] \underline{k} \cdot \underline{V}_1 = 0 \end{aligned} \right. \quad (4.11)$$

where

$$\left\{ \begin{aligned} A &= K T_0 (1 + Z_0 + N_0 Z_N) \\ B &= N_0 K (1 + Z_0 + T_0 Z_T) \end{aligned} \right. \quad (4.12)$$

Now from (4.9)

$$N_1 = \frac{N_0}{\omega} \underline{k} \cdot \underline{V}_1 ,$$

so (4.10 and (4.11), become,

$$- \omega \underline{V}_1 + \frac{A}{M\omega} \underline{k} (\underline{k} \cdot \underline{V}_1) + \underline{k} \frac{B T_1}{MN_0} = 0 \quad (4.13)$$

$$\left\{ \begin{aligned} & \underline{k} \cdot \underline{V}_1 \left\{ - N_0 \left(\frac{3}{2} A + \epsilon_0 + N_0 \epsilon_N \right) + \frac{iP_{fiss,0}}{\omega} + \frac{5}{2} (1 + Z_0) N_0 K T_0 + N_0 \epsilon_0 \right\}_1 \\ & + T_1 \left\{ - \omega \left(\frac{3}{2} B + N_0 \epsilon_T \right) - ik^2 K_{R0} \right\}_2 = 0 \end{aligned} \right. \quad (4.14)$$

Eliminate T_1 from (4.13)

$$-\omega \underline{V}_1 + \frac{A}{M\omega} \underline{k} (\underline{k} \cdot \underline{V}_1) - \underline{k} \frac{B}{MN_0} (\underline{k} \cdot \underline{V}_1) \frac{\{ \}_{1}}{\{ \}_{2}} = 0 \quad (4.15)$$

The modes all have \underline{V}_1 along \underline{k} so that we can take $\underline{k} \cdot$ and cancel $\underline{k} \cdot \underline{V}_1$ thereby obtaining a dispersion relation

$$\begin{aligned} \left(-\omega + \frac{Ak^2}{M\omega} \right) \left\{ -\omega \left(\frac{3}{2} B + N_0 \epsilon_T \right) - ik^2 K_{R0} \right\}_2 - \frac{k^2 B}{MN_0} \left\{ -N_0 \left(\frac{3}{2} A + \epsilon_0 + N_0 \epsilon_N \right) \right. \\ \left. + \frac{iP_{fiss,0}}{\omega} + \frac{5}{2} (1 + Z_0) N_0 K T_0 + N_0 \epsilon_0 \right\}_1 = 0 \end{aligned} \quad (4.16)$$

This dispersion relation is a cubic in ω and can be written,

$$\begin{aligned} \omega^3 \left(\frac{3}{2} B + N_0 \epsilon_T \right) + \omega^2 ik^2 K_{R0} + \omega \left\{ -\frac{Ak^2}{M} N_0 \epsilon_T \right. \\ \left. + \frac{k^2 B}{M} (\epsilon_0 + N_0 \epsilon_N) - \frac{k^2 B}{MN_0} \left[\frac{5}{2} (1 + Z_0) N_0 K T_0 + N_0 \epsilon_0 \right] \right\} \\ - \left\{ \frac{Aik^4}{M} K_{R0} + k^2 B \frac{iP_{fiss,0}}{MN_0} \right\} = 0 \end{aligned} \quad (4.17)$$

Free Sound Wave Case

If $K_{R0} = 0$ and $P_{fiss,0} = 0$, (4.17) becomes the relation for a simple sound wave (except for the peculiarity of a changing degree of ionization in the U-plasma as the temperature fluctuates). In this limit we find

$$\omega = \pm k V_s \quad (4.18)$$

$$V_s = \frac{\left\{ \frac{A}{M} N_0 \epsilon_T - \frac{B}{M} (\epsilon_0 + N_0 \epsilon_N) + \frac{B}{M} \left[\frac{5}{2} K T_0 (1 + Z_0) + \epsilon_0 \right] \right\}^{\frac{1}{2}}}{\left\{ \frac{3}{2} B + N_0 \epsilon_T \right\}^{\frac{1}{2}}} \quad (4.19)$$

where V_s is the sound speed. Note also that $d\omega/dk = \omega/k$ in this case.

If we use A and B given by (4.12), the sound speed can be written in the form

$$V_s = \left(\frac{5KT_0}{3M} \right)^{\frac{1}{2}} \frac{\left\{ \frac{3}{2} (1 + Z_0) - \frac{3N_0 \epsilon_N}{5KT_0} + \frac{3\epsilon_T (1 + Z_0 + N_0 Z_N)}{5K(1 + Z_0 + T_0 Z_T)} \right\}^{\frac{1}{2}}}{\left\{ \frac{3}{2} + \frac{\epsilon_T}{K(1 + Z_0 + T_0 Z_T)} \right\}^{\frac{1}{2}}} \quad (4.20)$$

General Case including Fission and Radiation

Two fairly simple limits can be extracted from the cubic (4.17). The first corresponds to the case in which the competing effects of radiation and fission nearly balance. The result is a growth or damping of the wave that is small. Thus we can write

$$\omega = \omega_R + \omega_I \cong kV_s + \omega_I \quad (4.21)$$

in (4.17) and treat ω_I as small (this gives an exact condition for the critical wavenumber for which ω_I changes sign). The cubic can be rewritten

$$\left(\frac{3}{2} B + N_0 \epsilon_T \right) \omega (\omega^2 - k^2 V_s^2) + \omega^2 i k^2 K_{R0} - \frac{A i k^4 K_{R0}}{M} - \frac{k^2 B i P_{fiss,0}}{M N_0} = 0 \quad (4.22)$$

Linearizing in ω_I gives

$$\omega_I \equiv \frac{i \left(\frac{B P_{fiss,0}}{M N_0} - k^2 V_S^2 K_{R0} + \frac{A k^2 K_{R0}}{M} \right)}{3 V_S^2 \left(B + \frac{2 N_0}{3} \epsilon_T \right)} \quad (4.23)$$

The system is unstable if the numerator of (4.23) is positive. There are two cases for which this is theoretically possible, namely

$$\left\{ \begin{array}{l} |k| < k_{crit} = \left[\frac{B P_{fiss,0}}{K_{R0} M N_0 \left(V_S^2 - \frac{A}{M} \right)} \right]^{\frac{1}{2}} \text{ for } V_S^2 > \frac{A}{M} \\ |k| > 0 \text{ for } V_S^2 < \frac{A}{M} \end{array} \right. \quad (4.24)$$

The inequality $V_S^2 > A/M$ can be written

$$1 + Z_0 > \frac{N_0 \epsilon_N}{K T} + \frac{3}{2} N_0 Z_N \quad (4.25)$$

The first range of unstable waves with wavelengths greater than a critical wavelength is similar to that found by Becker and McNeil. The expression (4.24) for the critical wavenumber, k_{crit} , at which waves go from stable to unstable is exact, but the growth rate ω_I for marginally unstable waves is an approximation which is valid for $|\omega_I| \ll |k V_S|$. It will turn out that $V_S^2 > A/M$, i.e., the inequality (4.25) is satisfied, so that only the first part of (4.24) is of interest.

The second limit that is easily extracted from (4.17) is the small k result. We see from (4.17) that $\omega \rightarrow 0$ as $k \rightarrow 0$. If we assume for small k that $\omega \sim k^n$ ($n > 0$), the various terms in (4.17) are of order k^{3n} , k^{2+n} , k^{2n+2} , k^4 , k^2 . As $k \rightarrow 0$ only the first and last terms survive. Retaining only these two terms and noting $i^{1/3} = (\sqrt{3} + i)/2$, we find the small k limit,

$$\omega \cong \frac{(\sqrt{3} + i)}{2} k^{2/3} \left[\frac{2B P_{fiss,0}}{3MN(B + \frac{2N_0}{3} \epsilon_T)} \right]^{1/3} \quad (4.26)$$

In this case we note $d\omega_R/dk = (2/3) \omega_R/k \rightarrow \infty$ as $k \rightarrow 0$.

The results (4.23), (4.24), and (4.26) can be expressed in terms of our original variables by eliminating A and B using (4.12). They become,

$$\omega_I = \frac{i \left\{ (1 + Z_0 + T_0 Z_T) \frac{P_{fiss,0}}{MN_0} - \frac{k^2 K_{RO}}{N_0 K} \left[V_s^2 - \frac{KT_0}{M} (1 + Z_0 + N_0 Z_N) \right] \right\}}{3V_s^2 \left\{ 1 + Z_0 + T_0 Z_T + \frac{2\epsilon_T}{3K} \right\}} \quad (4.27)$$

$$\equiv \frac{i \left\{ k_{crit}^2 - k^2 \right\} K_{RO} \left[1 - \frac{KT_0}{MV_s^2} (1 + Z_0 + N_0 Z_N) \right]}{3 N_0 K \left[1 + Z_0 + T_0 Z_T + \frac{\epsilon_T}{K} \right]}$$

(marginal instability)

$$\omega = \omega_R + i\omega_I = \frac{(\sqrt{3} + i)}{2} k^{2/3} \left\{ \frac{\frac{2}{3} \frac{P_{fiss,0}}{MN_0}}{1 + \frac{\epsilon_T}{2K(1 + Z_0 + T_0 Z_T)}} \right\}^{1/3} \quad (4.28)$$

(small k)

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$$k_{\text{crit}} = \left\{ \frac{K (1 + Z_0 + T_0 Z_T) \frac{P_{\text{fiss},0}}{M}}{K_{R0} \left[V_s^2 - \frac{KT_0}{M} (1 + Z_0 + N_0 Z_N) \right]} \right\}^{\frac{1}{2}} \quad (4.29)$$

(exact)

The behavior of $|\omega_I|$ is shown below in Fig. 4.1. The maximum growth rate is approximately the result obtained by setting $k = 0$ in (4.27).

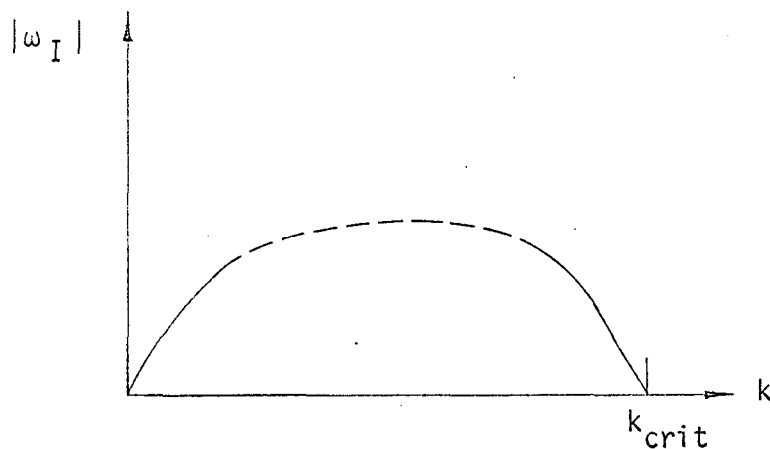


Figure 4.1. Schematic plot of the growth rate of unstable sound waves in a fissioning U-plasma as a function of wavenumber.

5. NONLINEAR BEHAVIOR OF UNSTABLE WAVES AND THEIR LIMITING AMPLITUDE IN A U-PLASMA SLAB

Linear analysis shows that a range of waves with wavenumbers $k < k_{\text{crit}}$ are unstable. The next question to answer is the limiting amplitude reached as a result of such instability. The principal damping mechanism that finally limits the growth of the waves is their mode coupling to that part of the spectrum that is radiation damped. This mode coupling should produce a steady spectrum of oscillations in a U-plasma, and in this section we shall compute this for a slab of plasma of thickness L occupying the space $0 < x < L$.

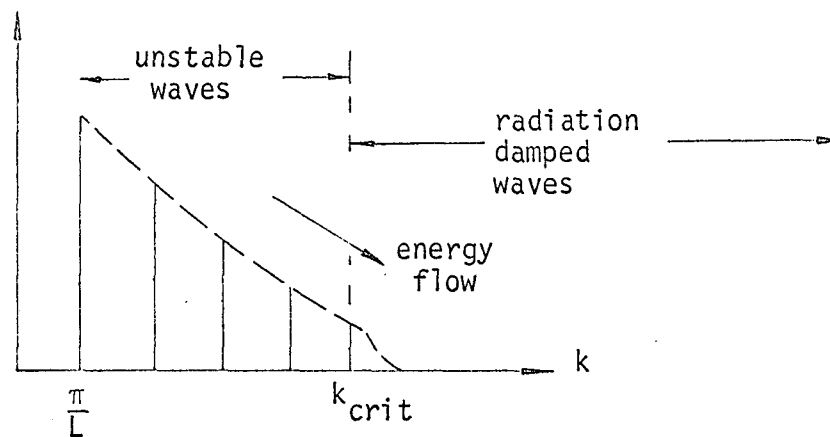


Figure 5.1.

In order to derive equations for the nonlinear wave amplitudes the multiple-time-scale form of perturbation theory⁽⁷⁾ will be used (which is similar to the Bogoliubov-Krylov-Mitropolsky method) through terms of second order in wave amplitudes. Thus we express the fluid variables as

$$\left\{ \begin{array}{l} N = N_0 + \epsilon N_1 \\ \underline{V} = \underline{0} + \epsilon \underline{V}_1 \\ T = T_0 + \epsilon T_1 \\ Z = Z_0 + \epsilon (N_1 Z_N + T_1 Z_T) \\ \epsilon = \epsilon_0 + \epsilon (N_1 \epsilon_N + T_1 \epsilon_T) \end{array} \right. \quad (5.1)$$

and attempt to find an asymptotic solution correct to $O(\epsilon^2)$. The parameter ϵ is introduced for bookkeeping purposes and is set equal to unity later. The problem is treated as one-dimensional. Second-order derivatives ϵ_{NN} , Z_{NN} , Z_{NT} , *etc.* will be neglected.

Zeroth-Order System

The steady-state slab of U-plasma must satisfy the zeroth-order equations

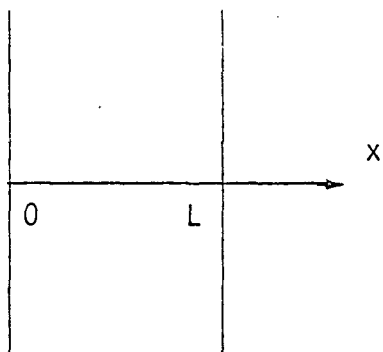


Figure 5.2

$$N_0 K T_0 (1 + Z_0) = \text{const.} = P_0, \quad (5.2)$$

and

$$P_{\text{fiss},0} + \frac{\partial}{\partial x} \left(K_{R0} \frac{\partial T_0}{\partial x} \right) = 0 \quad (5.3)$$

The first complication is now apparent, namely that the zeroth-order slab is necessarily inhomogeneous if $P_{\text{fiss},0} \neq 0$. The effect of inhomogeneity will be neglected in this analysis. It will be treated at a later date.

Equations to $O(\epsilon^2)$

Substituting (1) into the exact fluid equations and retaining terms of $O(\epsilon^2)$ leads to a system that can be written as,

$$\frac{\partial N_1}{\partial t} + N_0 \frac{\partial V_1}{\partial x} = \epsilon \psi \quad (5.4)$$

$$\frac{\partial V_1}{\partial t} + a \frac{\partial N_1}{\partial x} + b \frac{\partial T_1}{\partial x} = \epsilon \phi \quad (5.5)$$

$$c \frac{\partial N_1}{\partial t} + d \frac{\partial T_1}{\partial t} + g \frac{\partial V_1}{\partial x} = \epsilon \Lambda + \epsilon \left(e N_1 + K_{R0} \frac{\partial^2 T_1}{\partial x^2} \right) \quad (5.6)$$

where we have chosen to treat the combined effect of fission and radiation on the wave as $O(\epsilon)$ and have neglected the inhomogeneity of N_0 , T_0 , etc., and defined

$$\left\{ \begin{aligned}
 a &= \frac{KT_0}{MN_0} (1 + Z_0 + N_0 Z_N), \quad b = \frac{K}{M} (1 + Z_0 + T_0 Z_T) \\
 c &= \left[\frac{3}{2} KT_0 (1 + Z_0 + N_0 Z_N) + \epsilon_0 + N_0 \epsilon_N \right] \\
 d &= \left[\frac{3}{2} N_0 K (1 + Z_0 + T_0 Z_T) + N_0 \epsilon_T \right] \\
 e &= P_{\text{fiss},0}/N_0 \\
 g &= \frac{5}{2} (1 + Z_0) N_0 KT_0 + N_0 \epsilon_0
 \end{aligned} \right. \quad (5.7)$$

The quadratic terms ψ , ϕ , Λ on the right side of (5.4)-(5.6) are given by

$$\psi = - \frac{\partial}{\partial x} (N_1 V_1) \quad (5.8)$$

$$\begin{aligned}
 \phi &= - \frac{1}{MN_0} \frac{\partial}{\partial x} \left[N_1 KT_1 (1 + Z_0) + (N_0 KT_1 + N_1 KT_0) (N_1 Z_N + T_1 Z_T) \right] \\
 &+ \frac{N_1}{MN_0^2} \frac{\partial}{\partial x} \left[N_1 KT_0 (1 + Z_0 + N_0 Z_N) + N_0 KT_1 (1 + Z_0 + T_0 Z_T) \right] \\
 &- V_1 \frac{\partial V_1}{\partial x}
 \end{aligned} \quad (5.9)$$

$$\begin{aligned}
\Lambda = & - \frac{\partial}{\partial t} \left[\frac{3}{2} N_1 K T_1 (1 + Z_0) + \frac{3}{2} N_1 K T_0 (T_1 Z_T + N_1 Z_N) \right. \\
& + \left. \frac{3}{2} N_0 K T_1 (T_1 Z_T + N_1 Z_N) + N_1 (T_1 \epsilon_T + N_1 \epsilon_N) \right] \\
& - V_1 \frac{\partial}{\partial x} \left[\frac{3}{2} (1 + Z_0) K (N_1 T_0 + N_0 T_1) + \frac{3}{2} N_0 K T_0 (N_1 Z_N' + T_1 Z_T) \right. \\
& + \left. N_1 \epsilon_0 + N_0 (N_1 \epsilon_N + T_1 \epsilon_T) \right] \\
& - \frac{5}{2} (N_1 Z_N + T_1 Z_T) N_0 K T_0 \frac{\partial V_1}{\partial x} - \frac{5}{2} (1 + Z_0) (N_1 K T_0 + N_0 K T_1) \frac{\partial V_1}{\partial x} \\
& - N_0 (N_1 \epsilon_N + T_1 \epsilon_T) \frac{\partial V_1}{\partial x} - N_1 \epsilon_0 \frac{\partial V_1}{\partial x} .
\end{aligned} \tag{5.10}$$

Next take $\partial/\partial t$ of (5.5) and eliminate \dot{N}_1 using (5.4) in both (5.5) and (5.6):

$$\ddot{V}_1 + a \frac{\partial}{\partial x} \left(\epsilon \psi - N_0 \frac{\partial V_1}{\partial x} \right) + \frac{\partial \dot{T}_1}{\partial x} = \epsilon \dot{\phi} \tag{5.11}$$

$$c \left(\epsilon \psi - N_0 \frac{\partial V_1}{\partial x} \right) + d \dot{T}_1 + g \frac{\partial V_1}{\partial x} = \epsilon \Lambda + \epsilon \left(e N_1 + K_{RO} \frac{\partial^2 T_1}{\partial x^2} \right) . \tag{5.12}$$

Using (5.12) to eliminate \dot{T}_1 from (5.11) then yields the basic nonlinear fission-driven wave equation,

$$\begin{aligned}
 \frac{\partial^2 V_1}{\partial t^2} - V_s^2 \frac{\partial^2 V_1}{\partial x^2} &= \overbrace{\epsilon \frac{\partial \psi}{\partial x} \left(\frac{bc}{d} - a \right)}^{\text{mode coupling}} - \epsilon \frac{b}{d} \frac{\partial \Lambda}{\partial x} + \epsilon \frac{\partial \phi}{\partial t} \\
 &- \epsilon \frac{b}{d} \frac{\partial}{\partial x} \left(N_1 e + K_{R0} \frac{\partial^2 T_1}{\partial x^2} \right) \\
 &\quad \quad \quad \text{(fission) (radiation)}
 \end{aligned}
 \tag{5.13}$$

where the sound speed V_s is

$$V_s = \left\{ (g - c N_0) \frac{b}{d} + a N_0 \right\}^{\frac{1}{2}}
 \tag{5.14}$$

which readily reduces to equation (4.20) in section 4.

Now in lowest order (superscripts (o)) the general solution to (5.13) for the slab (which satisfies the boundary condition $V_1 = 0$ at $x = 0, L$) is

$$V_1^{(o)} = \sum_{n=-\infty}^{\infty} v_n e^{i\omega_n t} \sin k_n x
 \tag{5.15}$$

$$\begin{pmatrix} N_1^{(o)} \\ T_1^{(o)} \end{pmatrix} = \sum_{n=-\infty}^{\infty} \begin{pmatrix} N_n \\ T_n \end{pmatrix} e^{i\omega_n t} \cos k_n x, \quad n \neq 0,
 \tag{5.16}$$

where

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$$\left\{ \begin{array}{l} k_n = \frac{\pi n}{L}, \quad \omega_n = k_n V_s = \frac{\pi n V_s}{L} \\ v_{-n} = -v_n^*, \quad N_{-n} = N_n^*, \quad T_{-n} = T_n^* \end{array} \right. \quad (5.17)$$

and

$$N_n = \frac{i N_0}{V_s} v_n \quad (5.18)$$

$$T_n = i v_n \frac{V_s}{b} \left(1 - \frac{a N_0}{V_s^2} \right) \quad (5.19)$$

We must now derive equations for the complex amplitudes v_n through $O(\epsilon)$.

Multiple Time Scale Analysis

Our objective is to find an asymptotic solution to (5.13) correct through $O(\epsilon)$ and one that remains so through times of $O(\epsilon^{-1})$. To do this we use a multiple time scale form of perturbation theory⁽⁷⁾. Writing (5.13) as

$$\frac{\partial^2 v_1}{\partial t^2} - V_s^2 \frac{\partial^2 v_1}{\partial x^2} = \epsilon L, \quad (5.20)$$

we seek a solution of the form

$$v_1 = v_1^{(0)}(t, \epsilon t, \epsilon^2 t, \dots) + \epsilon v_1^{(1)}(t, \epsilon t, \dots) + \dots, \quad (5.21)$$

where $v_1^{(0)}$ is chosen so that it reduces to (5.15) as $\epsilon \rightarrow 0$. Similar expansions are made for N_1 and T_1 and the boundary conditions must be satisfied through all orders.

Two-Mode System

The critical wavelength, $2\pi/k_{\text{crit}}$, above which instability occurs is fairly large so that it is expected that many systems will contain at most only a few unstable modes. In order to simplify the analysis we shall now treat the case in which only the fundamental, $k_1 = \pi/L$, is unstable and the second harmonic, $k_2 = 2\pi/L$, is radiation-damped. Since higher harmonics ($n \geq 3$) are progressively more heavily damped than the $n = 2$ mode we can neglect them and approximate the system by a 2-mode system with one mode unstable and one mode damped.

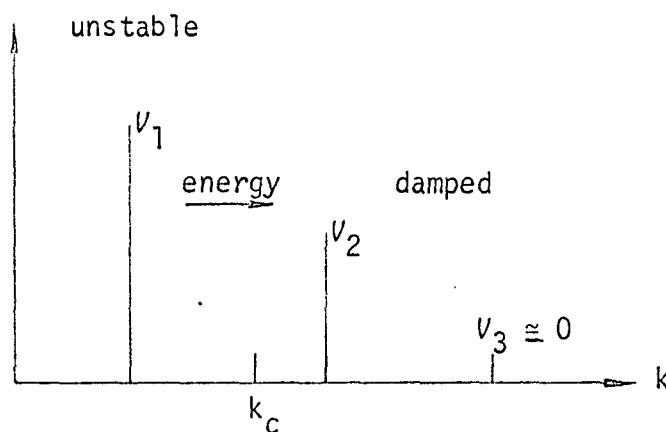


Figure 5.3

In this case

$$v_1^{(0)} = \sum_{n=-2}^2 v_n e^{i\omega_n t} \sin k_n x \quad (5.22)$$

and to $O(\epsilon)$ equation (5.20) becomes

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$$2 \frac{\partial}{\partial t} \frac{\partial}{\partial (\epsilon t)} V_1^{(0)} + \left(\frac{\partial^2}{\partial t^2} - V_s^2 \frac{\partial^2}{\partial x^2} \right) V_1^{(1)} = L^{(0)} \quad (5.23)$$

where $L^{(0)} = L(N_1^{(0)}, V_1^{(0)}, T_1^{(0)})$. We must use the ϵt dependence of $V_1^{(0)}$ to cancel those components of $L^{(0)}$ that would solve $(\partial^2/\partial t^2 - V_s^2 \partial^2/\partial x^2) L^{(0)}_{\text{resonant parts}} = 0$. These components, unless canceled, would give unwanted secular (t or x -proportional) terms in $V_1^{(1)}$.

Equating the first term on the left of (5.23) to the resonant terms of $L^{(0)}$ we have

$$2i \sum \omega_n \frac{\partial V_n}{\partial \epsilon t} e^{i\omega_n t} \sin k_n x = \left[\begin{array}{l} \text{Terms of } L^{(0)} \text{ of the type} \\ \sin(k_{1,2} x) e^{\pm i\omega_{1,2} t} \end{array} \right] \quad (5.24)$$

where

$$L = \left(\frac{bc}{d} - a \right) \frac{\partial \psi}{\partial x} - \frac{b}{d} \frac{\partial \Lambda}{\partial x} + \frac{\partial \phi}{\partial t} - \frac{b}{d} \frac{\partial}{\partial x} \left[N_1 e + K_{RO} \frac{\partial^2 T_1}{\partial x^2} \right] \quad (5.25)$$

Evaluation of Resonant Terms in Eq. (5.24).

By inspection of (5.25) and (5.8) - (5.10) for ψ , ϕ , Λ , we see that the following terms must be evaluated:

$$\frac{\partial^2}{\partial x^2} (N_1 V_1), \quad \frac{\partial^2}{\partial x \partial t} (N_1 T_1), \quad \frac{\partial^2}{\partial x \partial t} T_1^2$$

$$\frac{\partial}{\partial t} \left(N_1 \frac{\partial N_1}{\partial x} \right), \quad \frac{\partial}{\partial t} \left(N_1 \frac{\partial T_1}{\partial x} \right), \quad \frac{\partial}{\partial t} \left(V_1 \frac{\partial V_1}{\partial x} \right)$$

(5.26)

$$\frac{\partial}{\partial x} \left(v_1 \frac{\partial N_1}{\partial x} \right), \quad \frac{\partial}{\partial x} \left(v_1 \frac{\partial T_1}{\partial x} \right)$$

$$\frac{\partial}{\partial x} \left(T_1 \frac{\partial v_1}{\partial x} \right), \quad \frac{\partial}{\partial x} \left(N_1 \frac{\partial v_1}{\partial x} \right)$$

where the small density derivatives Z_N, ϵ_N will be neglected.*

The resonant terms will be extracted from the first of these in detail and the result for the rest simply listed since they all involve similar calculations.

Thus,

$$\frac{\partial^2}{\partial x^2} (N_1 v_1) = \frac{\partial^2}{\partial x^2} \sum_{n,m=-2}^2 N_n v_m e^{i(\omega_n + \omega_m)t} \cos k_n x \sin k_m x$$

Noting $\cos x \sin y = [\sin(x+y) - \sin(x-y)]/2$ this becomes,

$$= -\frac{1}{2} \sum N_n v_m e^{i(\omega_n + \omega_m)t} [(k_n + k_m)^2 \sin(k_n + k_m)x - (k_n - k_m)^2 \sin(k_n - k_m)x]$$

contributions to $\sin k_1 x$ $\left\{ \begin{array}{ll} n=2, m=-1 \\ n=-1, m=2 \\ n=-2, m=1 \\ n=1, m=-2 \end{array} \right.$ $\left\{ \begin{array}{ll} n=2, m=1 \\ \cdot \\ \cdot \\ \cdot \end{array} \right.$ $\left. \begin{array}{l} \sin k_1 x e^{3i\omega_1 t} \\ \text{nonresonant.} \end{array} \right\}$

*Typically (see section 6) $T\epsilon_T \sim 12 \cdot 10^{-11}$, $N\epsilon_N \sim 2 \cdot 10^{-11}$,
 $TZ_T \sim 2.6$, $NZ_N \sim 0.26$.

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$$\begin{array}{l} \text{contribution} \\ \text{to } \sin k_2 x \end{array} \left\{ \begin{array}{ll} n = 1, m = 1 & n = 1, m = -1 \\ n = -1, m = -1 & \cdot \\ & \cdot \\ & \cdot \end{array} \right\} \begin{array}{l} \sin k_2 x e^{0t} \\ \text{nonresonant} \end{array}$$

Extracting the resonant terms leads to,

$$\begin{aligned} \frac{\partial^2}{\partial x^2} (N_1 V_1) &\rightarrow -\frac{1}{2} (N_2 V_{-1} + N_{-1} V_2) e^{i\omega_1 t} k_1^2 \sin k_1 x \\ &+ \frac{1}{2} (N_{-2} V_1 + N_1 V_{-2}) k_1^2 \sin k_1 x e^{-i\omega_1 t} \\ &- \frac{1}{2} \left(N_1 V_1 e^{i\omega_2 t} - N_{-1} V_{-1} e^{-i\omega_2 t} \right) k_2^2 \sin k_2 x \\ &= -\frac{1}{2} (N_2 V_{-1} + N_{-1} V_2) e^{i\omega_1 t} k_1^2 \sin k_1 x - \frac{k_2^2}{2} N_1 V_1 e^{i\omega_2 t} \sin k_2 x + \text{c.c.} \end{aligned} \quad (5.27)$$

The remaining terms of (5.26) can be listed:

$$\begin{aligned} \frac{\partial^2}{\partial x \partial t} (N_1 T_1) &\rightarrow -\frac{i}{2} \omega_1 k_1 (N_2 T_{-1} + N_{-1} T_2) e^{i\omega_1 t} \sin k_1 x \\ &- \frac{i}{2} N_1 T_1 \omega_2 k_2 e^{i\omega_2 t} \sin k_2 x + \text{c.c.} \end{aligned} \quad (5.28)$$

$$\frac{\partial^2}{\partial x \partial t} (T_1^2) \rightarrow -i\omega_1 k_1 T_2 T_{-1} e^{i\omega_1 t} \sin k_1 x - \frac{i}{2} T_1^2 \omega_2 k_2 e^{i\omega_2 t} \sin k_2 x \quad (5.29)$$

+ c.c.

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$$\begin{aligned} \frac{\partial}{\partial t} \left(N_1 \frac{\partial T_1}{\partial x} \right) \rightarrow & \frac{i}{2} (k_1 N_2 T_{-1} - k_2 N_{-1} T_2) \omega_1 e^{i\omega_1 t} \sin k_1 x \\ & - \frac{i}{2} N_1 T_1 e^{i\omega_2 t} k_1 \omega_2 \sin k_2 x + \text{c.c.} \end{aligned} \quad (5.30)$$

$$\begin{aligned} \frac{\partial}{\partial t} \left(N_1 \frac{\partial N_1}{\partial x} \right) \rightarrow & - \frac{i}{2} k_1 \omega_1 N_2 N_{-1} e^{i\omega_1 t} \sin k_1 x \\ & - \frac{i}{2} k_1 \omega_2 N_1^2 e^{i\omega_2 t} \sin k_2 x + \text{c.c.} \end{aligned} \quad (5.31)$$

$$\begin{aligned} \frac{\partial}{\partial t} \left(V_1 \frac{\partial V_1}{\partial x} \right) \rightarrow & \frac{ik_1 \omega_1}{2} V_2 V_{-1} e^{i\omega_1 t} \sin k_1 x + \frac{ik_1 \omega_2}{2} V_1^2 e^{i\omega_2 t} \sin k_2 x \\ & + \text{c.c.} \end{aligned} \quad (5.32)$$

$$\begin{aligned} \frac{\partial}{\partial x} \left(V_1 \frac{\partial N_1}{\partial x} \right) \rightarrow & \frac{k_1^2}{2} (V_2 N_{-1} - 2 V_{-1} N_2) e^{i\omega_1 t} \sin k_1 x \\ & - V_1 N_1 k_1^2 e^{i\omega_2 t} \sin k_2 x + \text{c.c.} \end{aligned} \quad (5.33)$$

$$\begin{aligned} \frac{\partial}{\partial x} \left(V_1 \frac{\partial T_1}{\partial x} \right) \rightarrow & \frac{k_1^2}{2} (V_2 T_{-1} - 2 V_{-1} T_2) e^{i\omega_1 t} \sin k_1 x \\ & - V_1 T_1 k_1^2 e^{i\omega_2 t} \sin k_2 x + \text{c.c.} \end{aligned} \quad (5.34)$$

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$$\begin{aligned} \frac{\partial}{\partial x} \left(T_1 \frac{\partial V_1}{\partial x} \right) \rightarrow \frac{k_1^2}{2} (T_2 V_{-1} - 2T_{-1} V_2) e^{i\omega_1 t} \sin k_1 x \\ - T_1 V_1 k_1^2 e^{i\omega_2 t} \sin k_2 x + \text{c.c.} \end{aligned} \quad (5.35)$$

$$\begin{aligned} \frac{\partial}{\partial x} \left(N_1 \frac{\partial V_1}{\partial x} \right) \rightarrow \frac{k_1^2}{2} (N_2 V_{-1} - 2N_{-1} V_2) e^{i\omega_1 t} \sin k_1 x \\ - N_1 V_1 k_1^2 e^{i\omega_2 t} \sin k_2 x + \text{c.c.} \end{aligned} \quad (5.36)$$

We also require for the last part of (5.13),

$$\begin{aligned} \frac{\partial}{\partial x} \left(N_1 e + K_{R0} \frac{\partial^2 T_1}{\partial x^2} \right) \rightarrow k_1 (-eN_1 + K_{R0} k_1^2 T_1) e^{i\omega_1 t} \sin k_1 x \\ + 2k_1 (-eN_2 + 4K_{R0} k_1^2 T_2) e^{i\omega_2 t} \sin k_2 x . \end{aligned} \quad (5.37)$$

The remaining algebra is tedious but straightforward. It consists of finding equations for $\partial V_1 / \partial \epsilon t$ and $\partial V_2 / \partial \epsilon t$ from (5.24) using (5.25) for L , (5.8) - (5.10) for ψ , Λ , ϕ , together with (5.27) - (5.36) and the relations (5.18) and (5.19). The resulting equations become,

$$\begin{cases} \frac{\partial V_1}{\partial \epsilon t} = \gamma_1 V_1 + \alpha V_2 V_{-1} \\ \frac{\partial V_2}{\partial \epsilon t} = -\gamma_2 V_2 + \alpha V_1^2 \end{cases} \quad (5.38)$$

where

$$\begin{aligned}
 \gamma_1 &= \frac{b}{2V_S d} \left[\frac{eN_0}{V_S} - k_1^2 K_{RO} \frac{V_S}{b} \left(1 - \frac{aN_0}{V_S^2} \right) \right] = \frac{K_{RO}}{2d} \left(1 - \frac{aN_0}{V_S^2} \right) (k_{\text{crit}}^2 - k_1^2) \\
 \gamma_2 &= -\frac{b}{2V_S d} \left[\frac{eN_0}{V_S} - 4 k_1^2 K_{RO} \frac{V_S}{b} \left(1 - \frac{aN_0}{V_S^2} \right) \right] = -\frac{K_{RO}}{2d} \left(1 - \frac{aN_0}{V_S^2} \right) (k_{\text{crit}}^2 - 4k_1^2) \\
 \alpha &= \frac{k_1^2}{2\omega_1 d} \left\{ \frac{K}{2M} \left[\epsilon_0 T_0 Z_T + (1 + Z_0)(\epsilon_0 - T_0 \epsilon_T) \right] \frac{N_0}{V_S} \right. \\
 &\quad - K Z_T V_S^3 \frac{N_0 \epsilon_T}{M b^2} \left(1 - \frac{aN_0}{V_S^2} \right)^2 - \frac{V_S d}{2} - \frac{5}{4} N_0 V_S \left(1 - \frac{aN_0}{V_S^2} \right) \left(M b + \frac{2}{5} \epsilon_T \right) \\
 &\quad \left. - \left[\epsilon_0 + \frac{5}{2} (1 + Z_0) K T_0 \right] \frac{N_0 b}{2V_S} \right\}
 \end{aligned} \tag{5.40}$$

Nonlinear Behavior of Unstable Waves and their Limiting Amplitude

We now have a simple pair of equations describing the nonlinear evolution of an unstable wave of velocity amplitude V_1 . Equations (5.38) contain the destabilizing effect of the fission power density together with the stabilizing effects of radiation diffusion and mode coupling to the "nearest" damped mode V_2 in wavenumber space. The smallness parameter ϵ can now be set equal to unity and equations (5.38) for the wave amplitudes become

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$$\left\{ \begin{array}{l} \frac{\partial v_1}{\partial t} = \gamma_1 v_1 + \alpha v_2 v_{-1} \end{array} \right. \quad (5.41)$$

$$\left\{ \begin{array}{l} \frac{\partial v_2}{\partial t} = -\gamma_2 v_2 + \alpha v_1^2 \end{array} \right. \quad (5.42)$$

Now by adding v_1^* times (5.41) to v_1 times (5.41)* and noting from (5.17) that $v_{-1} = -v_1^*$ (and carrying out a similar operation for (5.42)) these equations become

$$\frac{\partial}{\partial t} |v_1|^2 = 2\gamma_1 |v_1|^2 - \alpha [v_1^{*2} v_2 + v_1^2 v_2^*] \quad (5.43)$$

$$\frac{\partial}{\partial t} |v_2|^2 = -2\gamma_2 |v_2|^2 + \alpha [v_2^* v_1^2 + v_2 v_1^{*2}]$$

Writing the complex amplitudes as $v_{1,2} = |v_{1,2}| e^{i\theta_{1,2}}$, these reduce further to

$$\frac{\partial}{\partial t} |v_1|^2 = 2\gamma_1 |v_1|^2 - 2\alpha |v_1|^2 |v_2| \cos(\theta_2 - 2\theta_1) \quad (5.44)$$

$$\frac{\partial}{\partial t} |v_2|^2 = -2\gamma_2 |v_2|^2 + 2\alpha |v_1|^2 |v_2| \cos(\theta_2 - 2\theta_1)$$

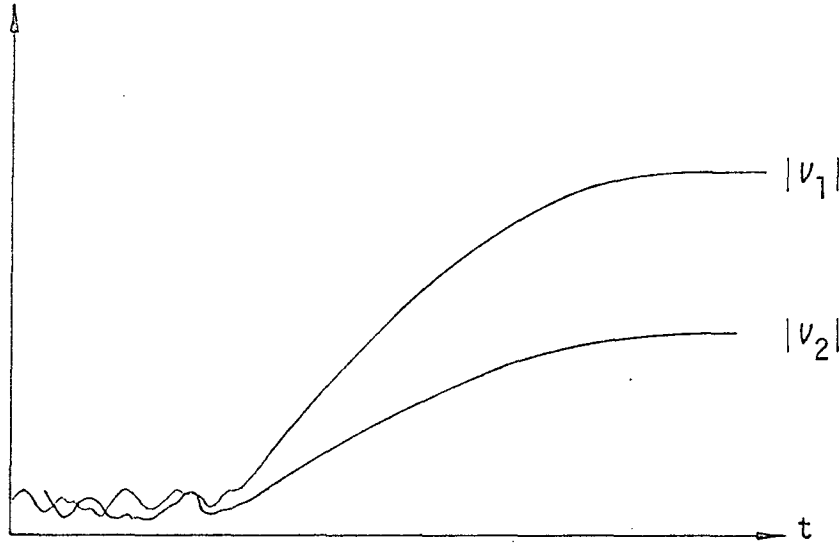


Figure 5.4

Suppose now a wave v_1 starts to grow out of the thermal noise and becomes of large amplitude. Since thermal fluctuations v_2 exist with all possible phases θ_2 , the mode v_2 which begins to grow due to mode coupling will be that for which the mode-coupling term in the v_2 equation is largest, *i.e.*, the mode v_2 with a phase such that $\cos(\theta_2 - 2\theta_1) = \pm 1$. The limiting amplitudes at which a steady state is reached follow from (5.44) as

$$\left\{ \begin{array}{l} |v_1|^2 = \frac{\gamma_1 \gamma_2}{\alpha^2} \\ |v_2| = \frac{\gamma_1}{-\alpha} \end{array} \right. \quad (5.45)$$

where γ_1 , γ_2 and α are given by (5.39) and (5.40), and the negative sign ($\cos(\theta_2 - 2\theta_1) = -1$) has been taken since α is negative.

Energy Density of a Nonlinear Wave

From (2.31) the averaged wave energy density follows as

$$\begin{aligned}
 W_{\text{wave}} = & \left\langle \frac{1}{2} M N_0 V_1^2 + \frac{3}{2} N_1 K T_1 (1 + Z_0) + \frac{3}{2} (N_1 K T_0 + N_0 K T_1) T_1 Z_T \right. \\
 & \left. + N_1 T_1 \epsilon_T \right\rangle
 \end{aligned}
 \tag{5.46}$$

$$\begin{aligned}
 \text{Now } V_1(\text{fundamental}) &= V_1 e^{i\omega_1 t} \sin k_1 x - V_{-1} e^{-i\omega_1 t} \sin k_1 x \\
 &= 2|V_1| \sin k_1 x \cos(\theta + \omega_1 t)
 \end{aligned}$$

where we used $V_{-1} = -V_1^*$ and set $V_1 = |V_1| e^{i\theta}$. Thus,

$$\begin{aligned}
 \langle V_1^2 \rangle &= |V_1|^2 \\
 \langle N_1 T_1 \rangle &= |V_1|^2 \frac{N_0}{b} \left(1 - \frac{aN_0}{V_s^2} \right) \\
 \langle T_1^2 \rangle &= |T_1|^2,
 \end{aligned}$$

where $\langle \rangle$ denotes a space and time average. Finally, using the relations (5.18) and (5.19) gives for the wave energy,

$$W_{\text{wave}} = \left\{ \frac{1}{2} M N_0 + \frac{d}{b} \left(1 - \frac{a N_0}{V_S^2} \right) + \frac{3}{2} N_0 K Z_T \frac{V_S^2}{b^2} \left(1 - \frac{a N_0}{V_S^2} \right)^2 \right\} |v_1|^2 \quad (5.47)$$

It is useful to compare this with the thermal energy and define the ratio

$$\begin{aligned} R_1 &= \frac{W_{\text{wave}}}{\frac{3}{2} N_0 K T_0 (1 + Z_0)} = \left(\frac{\text{wave energy}}{\text{thermal energy}} \right) \\ &= \frac{2 |v_1|^2}{3 N_0 K T_0 (1 + Z_0)} \left\{ \frac{1}{2} M N_0 + \frac{d}{b} \left(1 - \frac{a N_0}{V_S^2} \right) + \frac{3}{2} N_0 K Z_T \frac{V_S^2}{b^2} \left(1 - \frac{a N_0}{V_S^2} \right)^2 \right\} \end{aligned} \quad (5.48)$$

A similar expression exists for the second harmonic, etc.

In the next section some numerical examples will be given based on the various formulas derived in this and the previous section.

6. NUMERICAL CALCULATIONS USING PLASMA PARAMETERS CORRESPONDING TO THE NASA-LEWIS GAS-CORE DESIGN ENGINE

In this section we evaluate the various expressions for the growth rates, limiting amplitudes, etc. for a uranium plasma with characteristics corresponding to the NASA-Lewis gas-core design engine. The parameters were kindly supplied to us by R. Ragsdale of NASA-Lewis and are listed below:

$$\left\{ \begin{array}{ll} T & = 50,000 \text{ } ^\circ\text{K} \\ P_{\text{fiss}} & = 1.3 \cdot 10^{10} \text{ ergs/cm}^3\text{/sec} = \text{power density} \\ V_u & = 4.45 \cdot 10^6 \text{ cm}^3 = \text{U-plasma volume} \\ M_{\text{total U}} & = 2.7 \cdot 10^4 \text{ gm.} = \text{total U-mass in reactor} \\ \rho & = 6.07 \cdot 10^{-3} \text{ gm/cm}^3 = \text{U-density} \\ N_u & = 1.5 \cdot 10^{19} \text{ cm}^{-3} = \text{U number-density} \\ P_{\text{Tot}} & = 400 \text{ atmospheres} \cong 4 \cdot 10^8 \text{ ergs/cm}^3 = \text{pressure.} \end{array} \right. \quad (6.1)$$

For the Rosseland opacity we use the calculated data of Parks et al⁽⁸⁾. The mean Rosseland opacity is

$$k_R = 2 \cdot 10^4 \text{ cm}^2\text{/gm. for } P_{\text{tot}} = 4 \cdot 10^8, T = 5 \cdot 10^4 \text{ } ^\circ\text{K}$$

i.e., since $\rho = 6.07 \cdot 10^{-3} \text{ cm}^{-3}$ it becomes

$$k_R \cong 1.2 \cdot 10^2 \text{ cm}^{-1} \quad (6.2)$$

when expressed as an inverse length in the plasma.

The corresponding radiation diffusion coefficient follows as

$$K_R = \frac{16\sigma T^3}{3k_R} \cong 3.2 \cdot 10^8 \text{ erg cm}^{-1} \text{ sec}^{-1} \text{ deg}^{-1} \quad (6.3)$$

where σ is the black-body constant.

Evaluation of mean Charge, Z , mean Ionization Energy, ϵ , and their Partial Derivatives ϵ_T , Z_T , ϵ_N , Z_N .

The following quantities can be read off directly from the UPLAZ-2 graphs shown in Figs. 3.1 and 3.2 for a plasma at 50,000 °K and pressure of 400 Atm.:

$$\left\{ \begin{array}{ll} \epsilon & = 48 \text{ eV} \equiv 7.7 \cdot 10^{-11} \text{ ergs} \\ Z & = 3.8 \\ \epsilon_T & = 2.4 \cdot 10^{-15} \text{ ergs/deg.} \\ Z_T & = 5.37 \cdot 10^{-5} \text{ deg}^{-1} \\ T\epsilon_T & = 1.2 \cdot 10^{-10} \text{ ergs} \\ TZ_T & = 2.68 \end{array} \right. \quad (6.4)$$

The derivatives Z_N ($\equiv \partial Z / \partial N$) and ϵ_N are also of interest. They involve slightly more calculation since the graphs in Figs. 3.1 and 3.2 give Z , ϵ as functions of P , T instead of N , T . The conversion can be made by noting

$$\begin{aligned} \frac{\partial Z}{\partial N} &= \frac{\partial}{\partial N} Z (NKT(1+Z), T) \\ &= \frac{\partial Z}{\partial P} \left[KT(1+Z) + NKT \frac{\partial Z}{\partial N} \right], \end{aligned}$$

from which it follows that

$$\left\{ \begin{array}{l} N Z_N = \frac{N \frac{\partial Z}{\partial P} (1+Z) KT}{\left[1 - NKT \frac{\partial Z}{\partial P} \right]} = \frac{P \frac{\partial Z}{\partial P}}{\left[1 - \frac{P}{(1+Z)} \frac{\partial Z}{\partial P} \right]} \\ N \epsilon_N = P \frac{\partial \epsilon}{\partial P} \left[1 + \frac{P \frac{\partial Z}{\partial P}}{\left(1 + Z - P \frac{\partial Z}{\partial P} \right)} \right] \end{array} \right. \quad (6.5)$$

To obtain ϵ_p and Z_p we make use of graphs of $\epsilon(P)$ and $Z(P)$ shown in Figs. 6.1 and 6.2 which were constructed from Figs. 3.1 and 3.2. These gave

$$\begin{cases} P \epsilon_p = 12.6 \text{ eV} \equiv 2.02 \cdot 10^{-11} \text{ ergs} \\ P Z_p = .25 \end{cases} \quad (6.6)$$

so that

$$N Z_N = .264 \quad (6.7)$$

$$N \epsilon_N = 2.13 \cdot 10^{-11} \text{ ergs.}$$

We are now in a position to calculate various quantities of more direct physical interest.

Sound Speed

The sound speed as given by (4.20) is

$$V_s = \left(\frac{5KT_0}{3M} \right)^{\frac{1}{2}} \left\{ \frac{\frac{3}{2} (1 + Z_0) - \frac{3N_0\epsilon_N}{5KT_0} + \frac{3\epsilon_T (1 + Z_0 + N_0Z_N)}{5K (1 + Z_0 + T_0Z_T)}}{\frac{3}{2} + \frac{\epsilon_T}{K (1 + Z_0 + T_0Z_T)}} \right\}^{\frac{1}{2}} \quad (6.8)$$

Noting $KT_0 = 7 \cdot 10^{-12}$ ergs, and

$$\frac{KT}{M} = \frac{P}{\rho (1 + Z)} \quad , \quad (6.9)$$

Average ionization
energy, ϵ eV

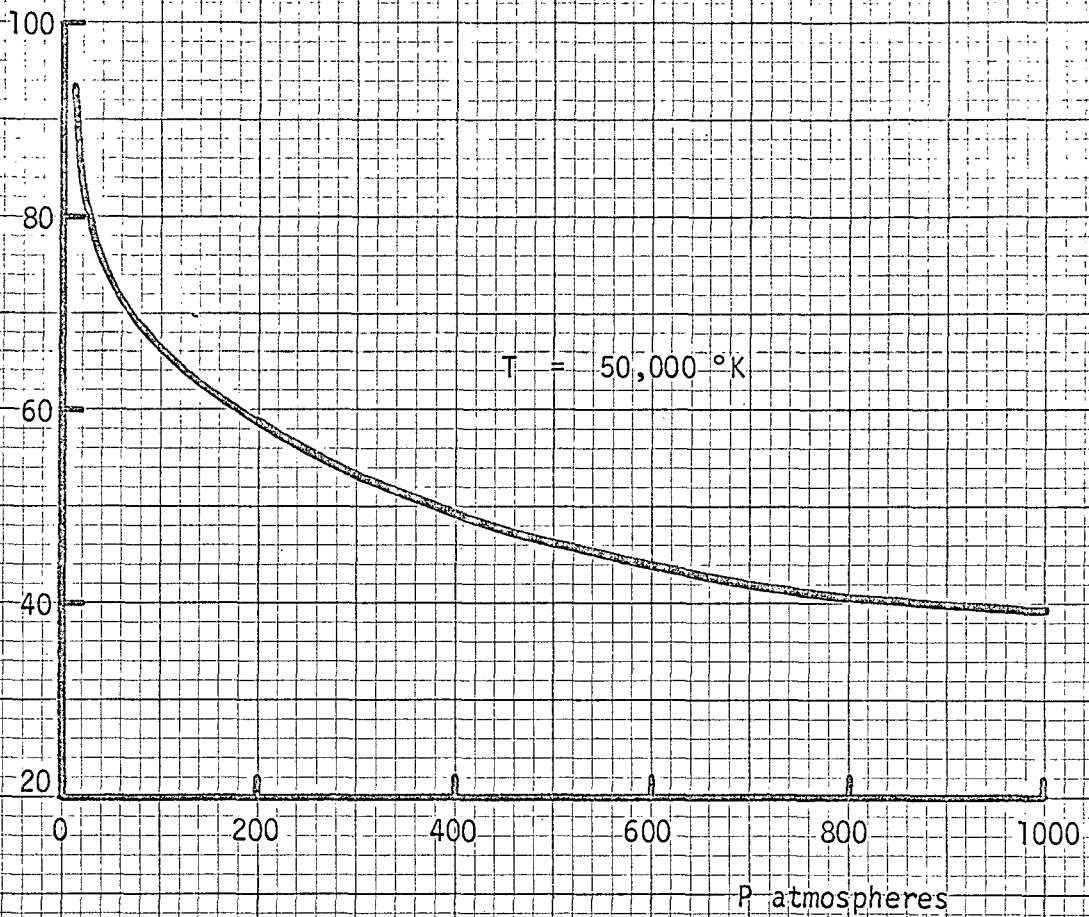


Figure 6.1

Average charge
number, Z .

5

4

3

0

200

400

600

800

1000

 $T = 50,000^{\circ}\text{K}$ P atmospheres

Figure 6.2

and using the values (6.1), (6.6), (6.7), we find

$$V_s \cong 2.73 \cdot 10^5 \text{ cm/sec.} \quad (6.10)$$

Critical Wavenumber

The critical wavenumber for instability is given by (4.29). To evaluate k_{crit} we need the quantity,

$$V_s^2 - \frac{A}{M} = \frac{\left(\frac{KT}{M} \right) \left\{ 1 + Z - \frac{N_o \epsilon_N}{KT} - \frac{3 N_o Z_N}{2} \right\}}{\left\{ \frac{3}{2} + \frac{\epsilon_T}{K(1 + Z_o + T_o Z_T)} \right\}} \quad (6.11)$$

It then follows from the above numerical values that

$$k_{\text{crit}} = \left\{ \frac{(1 + Z_o + T_o Z_T) P_{\text{fiss}} \left[\frac{3}{2} + \frac{\epsilon_T}{K(1 + Z_o + T_o Z_T)} \right]}{T K_{R0} \left[1 + Z - \frac{N_o \epsilon_N}{KT} - \frac{3 N_o Z_N}{2} \right]} \right\}^{\frac{1}{2}}$$

$$= \left\{ \frac{28.4 P_{\text{fiss}}}{1.35 T K_{R0}} \right\}^{\frac{1}{2}} \cong 1.32 \cdot 10^{-1} \quad (6.12)$$

The critical wavelength above which waves are unstable is thus,

$$\lambda_{\text{crit}} = \frac{2\pi}{k_{\text{crit}}} \sim 47.5 \text{ cm.} \quad (6.13)$$

Asymptotic Growth Rates

For small k the growth rate in (4.28) becomes

$$\omega_I \cong 4.1 \cdot 10^3 k^{2/3} \quad (6.14)$$

At the other limit, i.e., near marginal instability, we note

$$\omega_I \cong 1.25 \cdot 10^2 (k_{\text{crit}}^2 - k^2) \quad (6.15)$$

where we used $V_S^2 - A/M \cong 4.5 \cdot 10^9$.

An estimate of the maximum growth rate is

$$\omega_{I \text{ Max}} \cong 1.25 \cdot 10^2 k_{\text{crit}}^2 \cong 2.18 \text{ sec}^{-1} \quad (6.16)$$

These various values of ω_I are plotted in Fig. 6.3.

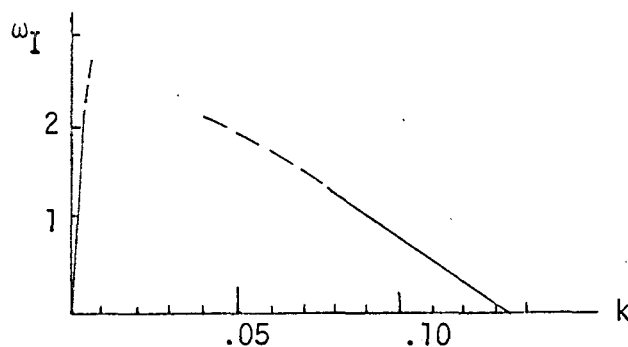


Figure 6.3.

Growth rate of unstable waves as a function of wavenumber.

We see that the growth rate is rather slow, *i.e.*, it would typically take a few seconds for the unstable waves to e-fold up in amplitude to their saturation value. This is to be compared with the real part of the frequency, which is of order (not exactly equal to) $k_{\text{crit}} V_s \cong 3.6 \cdot 10^4 \text{ sec}^{-1}$.

Nonlinear Amplitude

The limiting amplitude resulting from instability follows from (5.45) as

$$\frac{|v_1|}{V_s} = \frac{(\gamma_1 \gamma_2)^{\frac{1}{2}}}{|\alpha| V_s} = \frac{K_{RO} \left(1 - \frac{aN_0}{V_s^2}\right)}{|\{ \} |} \frac{1}{k_1} \left[(k_{\text{crit}}^2 - k_1^2)(4k_1^2 - k_{\text{crit}}^2) \right]^{\frac{1}{2}} \quad (6.17)$$

where

$$\begin{aligned} \{ \} = & \frac{K}{2M} \frac{N_0}{V_s} \left[\epsilon_0 T_0 Z_T + (1 + Z) (\epsilon_0 - T_0 \epsilon_T) \right] \\ & - K Z_T V_s^3 \frac{N_0 \epsilon_T}{Mb^2} \left(1 - \frac{aN_0}{V_s^2} \right)^2 - \frac{V_s d}{2} - \frac{5}{4} N_0 V_s \left(1 - \frac{aN_0}{V_s^2} \right) \left(Mb + \frac{2}{5} \epsilon_T \right) \\ & - \frac{N_0 b}{2V_s} \left[\epsilon_0 + \frac{5}{2} (1 + Z) K T_0 \right] \end{aligned}$$

We require the following quantities:

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$$\left\{ \begin{array}{l} [\epsilon_0 T_0 Z_T + (1 + Z)(\epsilon_0 - T_0 \epsilon_T)] \approx 0 \\ \frac{aN_0}{V_S^2} = .89, \left(1 - \frac{aN_0}{V_S^2}\right) = .11 \\ \frac{d}{N_0} = 3.97 \cdot 10^{-15} \\ Mb = 10.45 \cdot 10^{-16} \\ MV_S^2 = 2.92 \cdot 10^{-11} \end{array} \right.$$

Writing

$$\begin{aligned} \frac{1}{N_0 V_S} \{ \} &= \frac{K}{2 MV_S^2} [\epsilon_0 T_0 Z_T + (1 + Z)(\epsilon_0 - T_0 \epsilon_T)] \quad (1) \\ &= K Z_T \frac{MV_S^2 \epsilon_T}{M^2 b^2} \left(1 - \frac{aN_0}{V_S^2}\right)^2 - \frac{d}{2N_0} - \frac{5}{4} \left(1 - \frac{aN_0}{V_S^2}\right) (Mb + \frac{2}{5} \epsilon_T) \quad (2) \quad (3) \quad (4) \\ &= \frac{Mb}{2 MV_S^2} \left[\epsilon_0 + \frac{5}{2} (1 + Z) K T_0 \right] \quad (5) \end{aligned}$$

the various terms as labeled have values

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$$\left\{ \begin{array}{l} \textcircled{1} \cong 0 \\ \textcircled{2} \cong -5.83 \cdot 10^{-18} \\ \textcircled{3} \cong -1.98 \cdot 10^{-15} \\ \textcircled{4} \cong -2.77 \cdot 10^{-16} \\ \textcircled{5} \cong -2.88 \cdot 10^{-15} \end{array} \right.$$

Thus

$$\frac{1}{N_0 V_S} \{ \} \cong -5.14 \cdot 10^{-15} ,$$

from which it follows that

$$\boxed{\frac{|V_1|}{V_S} \cong 1.67 \cdot 10^{-3} \frac{1}{k_1} \left[(k_{\text{crit}}^2 - k_1^2)(4k_1^2 - k_{\text{crit}}^2) \right]^{\frac{1}{2}}} \quad (6.19)$$

Ratio of wave energy to Thermal Energy

This ratio, which is given by (5.48) is

$$R_1 = \frac{2}{3} \Gamma^2 \frac{N_0 M V_S^2}{P} \left[\frac{1}{2} + \frac{d}{N_0 M b} \left(1 - \frac{a N_0}{V_S^2} \right) + \frac{3}{2} K Z_T \frac{M V_S^2}{M^2 b^2} \left(1 - \frac{a N_0}{V_S^2} \right)^2 \right]$$

where for the plasma parameters listed in this section the terms in the

C-2

square bracket are

$$[] \cong .5 + .418 + .003 \cong .921 .$$

Thus

$$R_1 = \left(\frac{\text{wave energy}}{\text{thermal energy}} \right) = 0.675 \frac{|V_1|^2}{V_S^2} \quad (6.20)$$

where $|V_1|/V_S$ is given by (6.19).

Concluding Numerical Comments

The above formulas (6.19) and (6.20) apply for a slab in which a single mode of wavenumber k_1 is unstable due to coupling to a damped mode k_2 . Now recall that $k_1 = \pi/L$ must satisfy $k_1 < k_{\text{crit}} < k_2 = 2k_1$ in order for our two-mode analysis to be correct. Choosing for example $k_1 = 2k_{\text{crit}}/3$ (*i.e.*, corresponding to a plasma slab of thickness

$L \cong 36$ cm.) gives $|V_1|/V_S \sim 2 \cdot 10^{-4}$ and $R \sim 3 \cdot 10^{-8}$.

The waves that result from the instability are thus seen to be of negligible amplitude in this case. This is because they have a relatively slow growth rate ($\omega_I/kV_S \sim 10^4$) and lose their energy *via* mode coupling to the radiation damped modes. For a thicker slab of plasma several modes become unstable and the algebra of the analysis becomes more complicated. However this is unlikely to change the above conclusion provided only a few modes are unstable, *i.e.*, provided L does not exceed a couple of meters. For example one would expect the energy in the fundamental to increase as $L^{5/2}$ (assuming a Kolmogoroff spectrum) which would increase the pressure fluctuation level to $R \sim 10^{-5}$ for a slab of thickness 2.4 meters.

7. DEPENDENCE OF THE FISSION POWER DENSITY ON THE FLUID VELOCITY OF THE URANIUM PLASMA AND ITS ROLE IN THE INSTABILITY OF SOUND WAVES

The density of thermal neutrons within the reactor cavity consists principally of neutrons returned to the cavity from the surrounding moderator. Due to the open nozzle of the rocket chamber there will be an effective average drift of the neutron gas out of the nozzle along with the hydrogen propellant. For purposes of calculation we shall assume that the thermal neutron flux in the rocket chamber can be represented by a Maxwellian distribution,

$$f_n = \frac{N}{(2\pi v_n)^{3/2}} \exp - \frac{(\underline{v} - \underline{v}_n)^2}{2v_n^2} \quad (7.1)$$

where \underline{v}_n is the drift velocity towards the nozzle and v_n the neutron thermal velocity. The space dependence of N or \underline{v}_n will be neglected for the present, but should be discussed in a later study.

Similarly the uranium plasma can undergo fluid motions with velocity \underline{V} within the reactor cavity and the appropriate distribution function for the uranium nuclei is

$$f_u = \frac{N}{(2\pi v_u)^{3/2}} \exp - \frac{(\underline{v} - \underline{V})^2}{2v_u^2} \quad (7.2)$$

Now due to the dependence of the neutron-induced fission cross-section on the relative velocity between a neutron and uranium nucleus, a coupling can occur between the plasma motion and the fission power density. The purpose of the analysis in this section is to examine the consequences of this coupling for the question of instability.

The general expression for the fission power density can be written

$$P_{fiss} = \int \int d\underline{v} d\underline{v}' |\underline{v} - \underline{v}'| Q f_n(\underline{v}) f_u(\underline{v}') \sigma_a(|\underline{v} - \underline{v}'|) \quad (7.3)$$

where Q is the kinetic energy in fission fragments in the plasma rest-frame (recall that in the energy equation we took the moment using $\frac{1}{2} M(\underline{v} - \underline{v}')^2$ so that P_{fiss} is the rate of increase of thermal energy in the uranium plasma). The integral involves the Maxwellian distributions above in which it is of interest to note that the thermal velocities of the neutrons and uranium nuclei are comparable and the drift velocities \underline{v}_n and \underline{v} are subsonic.

The integral required depends on the neutron absorption cross-section σ_a . This cross-section has a large number of resonances and it is usual to model these by a single resonance level using the Breit-Wigner formula (see Blatt and Weisskopf⁽⁹⁾),

$$\sigma_a = \pi \lambda^2 g_J \frac{\Gamma_n \Gamma_\gamma}{(E_r - E_0)^2 + \left(\frac{\Gamma}{2}\right)^2} \quad (7.4)$$

where

E_0 = energy of resonance model for nucleus,

E_r = kinetic energy associated with relative velocity between neutron and nucleus

Γ_γ, Γ_n = radiative capture and neutron widths

and $\Gamma = \Gamma_\gamma + \Gamma_n$.

Now the energy E_0 of the lowest neutron resonance is about 1 eV. In this analysis we will assume that the thermal neutron temperature is below 10^4 °K in which case σ_a becomes approximately constant. This greatly simplifies the evaluation of (7.3) which then becomes,

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$$\begin{aligned} \frac{P_{fiss}}{\sigma_a Q} \frac{(2\pi v_n^2)^{3/2}}{N} \frac{(2\pi v_u^2)^{3/2}}{N} &= \int d\underline{v} d\underline{v}' |\underline{v} - \underline{v}'| e^{-\frac{(\underline{v}-\underline{v}_n)^2}{2v_n^2}} e^{-\frac{(\underline{v}'-\underline{v})^2}{2v_u^2}} \\ &= \int d\underline{v} d\underline{v}' |\underline{v} - \underline{v}' - \underline{v}_n + \underline{v}| e^{-\frac{v^2}{2v_n^2}} e^{-\frac{v'^2}{2v_u^2}} \end{aligned}$$

Noting

$$\int d\underline{v} |\underline{v} - \underline{A}| e^{-\frac{v^2}{2v_n^2}} = \frac{4\pi}{3} \int_0^\infty v dv (3v^2 + A^2) e^{-\frac{v^2}{2v_n^2}}$$

we find

$$P_{fiss} = N\sigma_a Q \frac{1}{(2\pi)^{3/2} v_n} \frac{4\pi}{3} \left\{ 6 v_n^2 + |\underline{v}_n - \underline{v}|^2 + 3v_u^2 \right\} \quad (7.5)$$

Since $v_n, v_u \gg |\underline{v}_n|, |\underline{v}|$ the expression for P_{fiss} can be written

$$P_{fiss} \cong P_{fiss,0} + \underline{v} \cdot \underline{P}'_{fiss,0} \quad (7.6)$$

where for the above assumed cross-section

$$\underline{P}'_{fiss,0} \equiv \left. \frac{\partial P_{fiss,0}}{\partial \underline{v}} \right|_{\underline{v}=0} = \frac{2\underline{v}_n P_{fiss,0}}{3(2v_n^2 + v_n^2)} \quad (7.7)$$

It should be emphasized that the general form (7.6) remains correct for the more complicated functional dependence of σ_a as given by (7.4), but $P'_{fiss,0}$ would become then more complicated. It is interesting to note that a linear term in \underline{V} only occurs if $\underline{V}_n \neq 0$, i.e., the neutron population is drifting through the uranium population. The instability that results (as will be seen in the next section) is analogous to the two-stream instability of longitudinal waves that occurs in plasmas for which relative drift between species is present.

Dispersion Relation for Sound Waves including the Velocity Dependence of the Fission Power Density.

The fluid equations and linearization proceed as in section 4. Retaining the extra term (last term of (7.6)) the linear system of equations becomes,

$$\frac{\partial N_1}{\partial t} + N_0 \frac{\partial}{\partial \underline{x}} \cdot \underline{V}_1 = 0 \quad (7.8)$$

$$\frac{\partial \underline{V}_1}{\partial t} + \frac{1}{MN_0} \frac{\partial}{\partial \underline{x}} \cdot \left[N_1 KT_0 (1 + Z_0 + N_0 Z_N) + N_0 KT_1 (1 + Z_0 + T_0 Z_T) \right] = 0 \quad (7.9)$$

$$\begin{aligned} & \frac{3}{2} \frac{\partial}{\partial t} \left[N_1 KT_0 (1 + Z_0 + N_0 Z_N) + N_0 KT_1 (1 + Z_0 + T_0 Z_T) \right] \\ & + \frac{\partial}{\partial t} \left[+ N_1 \epsilon_0 + N_0 N_1 \epsilon_N + N_0 T_1 \epsilon_T \right] + \frac{5}{2} (1 + Z_0) N_0 KT_0 \frac{\partial}{\partial \underline{x}} \cdot \underline{V}_1 \\ & + N_0 \epsilon_0 \frac{\partial}{\partial \underline{x}} \cdot \underline{V}_1 = \frac{N_1}{N_0} P_{fiss,0} + \underline{V}_1 \cdot \underline{P}'_{fiss,0} + K_{Ro} \nabla^2 T_1 \end{aligned} \quad (7.10)$$

As before we set

$$(N_1, \underline{V}_1, T_1) \rightarrow (N_1, \underline{V}_1, T_1) e^{i(\underline{k} \cdot \underline{x} - \omega t)}, \quad (7.11)$$

and eliminating N_1 the transformed equations become

$$-\omega \underline{V}_1 + \frac{A}{M\omega} \underline{k}(\underline{k} \cdot \underline{V}_1) + \underline{k} \frac{BT_1}{MN_0} = 0 \quad (7.12)$$

$$\begin{aligned} \underline{k} \cdot \underline{V}_1 \left\{ -N_0 \left(\frac{3}{2} A + \epsilon_0 + N_0 \epsilon_N \right) + \frac{i P_{fiss,0}}{\omega} + \frac{5}{2} (1 + Z_0) N_0 K T_0 + N_0 \epsilon_0 \right\}_1 \\ + i \underline{V}_1 \cdot \underline{P}'_{fiss,0} + T_1 \left\{ -\omega \left(\frac{3}{2} B + N_0 \epsilon_T \right) - i k^2 K_{R0} \right\}_2 = 0 \end{aligned} \quad (7.13)$$

Eliminating T_1 from (7.12),

$$-\omega \underline{V}_1 + \frac{A}{M\omega} \underline{k}(\underline{k} \cdot \underline{V}_1) - \underline{k} \frac{B}{MN_0} \left[\frac{\underline{k} \cdot \underline{V}_1 \left\{ \right\}_1 + i \underline{V}_1 \cdot \underline{P}'_{fiss,0}}{\left\{ \right\}_2} \right] = 0 \quad (7.14)$$

Now the waves are longitudinal, *i.e.*, \underline{k} is parallel or antiparallel to \underline{V}_1 . Choose \underline{V}_1 as x axis and set $\underline{k} \equiv (k, 0, 0)$ (k has a sign). Then

$$\underline{V}_1 \cdot \underline{P}'_{fiss,0} / \underline{k} \cdot \underline{V}_1 = P'_{fiss,0} \cos \theta / k, \text{ where } \theta \text{ is the angle between } \underline{V}_1$$

and $\underline{P}'_{fiss,0}$. Taking $\underline{k} \cdot$ of (7.14) then leads to the dispersion relation

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$$\begin{aligned} & \left(-\omega + \frac{Ak^2}{M\omega} \right) \left\{ -\omega \left(\frac{3}{2} B + N_0 \epsilon_T \right) - i k^2 K_{R0} \right\}_2 \\ & - \frac{k^2 B}{MN_0} \left\{ -N_0 \left(\frac{3}{2} A + \epsilon_0 + N_0 \epsilon_N \right) + \frac{i P_{fiss,0}}{\omega} + \frac{5}{2} (1 + Z_0) N_0 K T_0 + N_0 \epsilon_0 \right\}_1 \\ & - \frac{i B k}{MN_0} P'_{fiss,0} \cos \theta = 0 \end{aligned}$$

This is again a cubic in ω :

$$\begin{aligned} & \omega^3 \left(\frac{3}{2} B + N_0 \epsilon_T \right) + \omega^2 i k^2 K_{R0} + \omega \left\{ -\frac{AK^2}{M} N_0 \epsilon_T \right. \\ & \left. + N_0 \epsilon_N \frac{K^2 B}{M} - \frac{k^2 B}{M} \frac{5}{2} K T_0 (1 + Z_0) - \frac{i B k}{MN_0} P'_{fiss,0} \cos \theta \right\} \\ & - \left\{ \frac{A i k^4}{M} K_{R0} + \frac{k^2 B i}{MN_0} P_{fiss,0} \right\} = 0 \end{aligned} \quad (7.15)$$

The sound speed V_s can be defined as in section 4 and the cubic rewritten

$$\begin{aligned} & \left(\frac{3}{2} B + N_0 \epsilon_T \right) \omega \left(\omega^2 - k^2 V_s^2 \right) + \omega^2 i k^2 K_{R0} - i \omega \frac{B k}{MN_0} P'_{fiss,0} \cos \theta \\ & - \frac{A i k^4 K_{R0}}{M} - \frac{k^2 B i}{MN_0} P_{fiss,0} = 0 \end{aligned} \quad (7.16)$$

Marginal Instability Condition

For the case that the imaginary terms are small in (7.16) we can write

$$\omega = \omega_R + i\omega_I \cong kV_S + \omega_I \quad (7.17)$$

and derive the marginal instability growth rate by linearizing in ω_I . It follows from (7.16) that,

$$\omega_I \cong \frac{i \left[\frac{B}{MN_0} (P_{fiss,0} + V_S P'_{fiss,0} \cos \theta) - k^2 V_S^2 K_{Ro} + \frac{Ak^2 K_{Ro}}{M} \right]}{3 V_S^2 \left(B + \frac{2N_0 \epsilon_T}{3} \right)} \quad (7.18)$$

The unstable wavenumber range now becomes,

$$|k| < k_{crit} = \left\{ \frac{B (P_{fiss,0} + V_S P'_{fiss,0} \cos \theta)}{K_{Ro} MN_0 \left(V_S^2 - \frac{A}{M} \right)} \right\}^{\frac{1}{2}} \quad (7.19)$$

Small k Limit

In the small-k limit the velocity-dependence of P_{fiss} does not change the result found in section 4, so we again have

$$\omega \cong \frac{(\sqrt{3} + i)}{2} k^{2/3} \left[\frac{2 B P_{fiss,0}}{3 MN_0 \left(B + \frac{2N_0 \epsilon_T}{3} \right)} \right]^{1/3} \quad (7.20)$$

General Comments

The growth rate (7.18) near marginal instability behaves as if the fission power density was replaced by

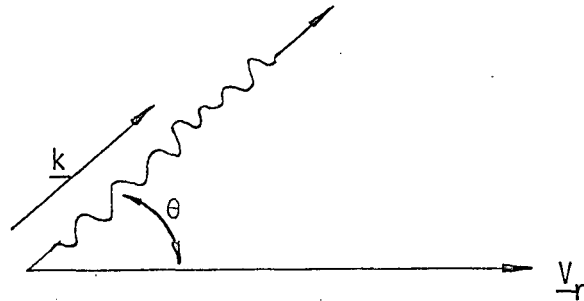
$$P_{fiss,0} + V_s P'_{fiss,0} \cos \theta$$

The velocity dependence of P_{fiss} thus has a destabilizing influence if the second term is positive and is stabilizing if it is negative.

Now from (7.7) we see that

$$V_s P'_{fiss,0} \cos \theta = \frac{2V_s V_n P_{fiss,0} \cos \theta}{3(2v_n^2 + v_u^2)} \quad (7.21)$$

Thus the forward cone of waves (along \underline{V}_n) is unstable with growth rates that are largest for $\theta = 0$. However, since $V_s \sim v_u$,



and $V_n < v_n$, the absolute change in growth rate for the unstable waves that derives from this effect is fairly small.

Effect of the fluid velocity dependence of the Fission power density on the Nonlinear behavior of Unstable Waves

In this section we carry through the extra term, $\underline{V} \cdot P'_{fiss,0}$,

in the nonlinear multiple-time scale analysis of section 5. The nonlinear sound wave equation (5.13) then becomes

$$\begin{aligned} \frac{\partial^2 V_1}{\partial t^2} - V_s^2 \frac{\partial^2 V_1}{\partial x^2} &= \overbrace{\epsilon \frac{\partial \psi}{\partial x} \left(\frac{bc}{d} - a \right)}^{\text{mode coupling}} - \epsilon \frac{b}{d} \frac{\partial \Lambda}{\partial x} + \epsilon \frac{\partial \phi}{\partial t} \\ &- \epsilon \frac{b}{d} \frac{\partial}{\partial x} \left\{ \underbrace{N_1 e}_{\text{(fission)}} + \underbrace{V_1 P'_{\text{fiss},0}}_{\substack{\text{(fission} \\ \text{Doppler} \\ \text{effect)}}} + \underbrace{K_{Ro} \frac{\partial^2 T_1}{\partial x^2}}_{\text{(radiation)}} \right\}. \end{aligned} \quad (7.22)$$

If we make the same expansion as before, with a zeroth-order expression for V_1 ,

$$V_1^{(0)} = \sum_{n=-\infty}^{\infty} v_n e^{i\omega_n t} \sin k_n x, \quad (7.23)$$

it turns out that terms of the form $\sin k_1 x e^{i\omega_1 t}$ occur in $O(\epsilon)$ throughout (7.22), except for the extra term involving $P'_{\text{fiss},0}$. This extra term only contains first harmonic components of the form $\cos k_1 x e^{i\omega_1 t}$. Multiplying the $O(\epsilon)$ equation by $\sin k_1 x$ and performing $\int_0^L dx$, the effect of the $P'_{\text{fiss},0}$ term therefore averages to zero, *i.e.*, for a bounded geometry it has no effect on the nonlinear growth of waves.

The reason for this can best be seen by including the $P'_{\text{fiss},0}$ term in the zeroth-order equation and considering the equation

$$\frac{\partial^2 V_1}{\partial t^2} - V_s^2 \frac{\partial^2 V_1}{\partial x^2} + \alpha \frac{\partial V_1}{\partial x} = 0, \quad (7.24)$$

where $\alpha = b P'_{fiss,0}/d$, for the slab geometry. Substituting traveling wave solutions $e^{i(\omega t - kx)}$ gives a dispersion relation

$$\omega = \pm (k^2 V^2 - i\alpha k)^{\frac{1}{2}} \cong \pm kV \left(1 - \frac{i\alpha k}{2k^2 V^2} \right), \quad (7.25)$$

so that a solution of wavenumber k satisfying (7.24) is

$$V_1 \cong A e^{\frac{\alpha t}{2V}} e^{i(kVt - kx)} + B e^{-\frac{\alpha t}{2V}} e^{-i(kVt + kx)} \quad (7.26)$$

It is clear from (7.26) that the right traveling wave is growing in amplitude ($\alpha > 0$), and the left-traveling wave decaying. Now for a bounded geometry with perfectly reflecting boundaries a standing wave consists of a superposition of a right and left traveling wave. As the wave traverses the slab thickness L to the right it grows in amplitude. Then, after reflection, it decays an equal amount on traveling to the left. The average effect is a zero growth for a standing wave in a slab.

In the actual case of a gas core nuclear rocket engine, the only waves which can propagate freely out of the U-plasma without reflection, are those propagating out of the nozzle. These will undergo at most only a few e-foldings in the course of their transit across the U-plasma and are not expected to be of importance.

8. COMMENTS ON THERMAL FLUCTUATIONS IN A NEUTRON GAS AND IN THE DISTRIBUTION OF FISSION EVENTS IN A U-PLASMA

The main point of the study described in this report is to calculate the behavior of those unstable waves in the system that may become of large amplitude. However it is worth noting that there is of course an irreducible thermal fluctuation level in such a plasma. These thermal fluctuations provide the initial amplitude from which unstable waves may grow. Fluctuations in the fission power density, although small, will be larger than is normally the case due to the large amount of energy (200 MeV) released per fission event.

Consider first the fluctuations in the noninteracting neutron gas of N neutrons in a box of volume V . The neutron distribution function can be written

$$F = \frac{1}{N!} \sum_{i=1}^N \delta[\underline{x} - \underline{x}_i(t)] \delta(\underline{v} - \underline{v}_i) \quad (8.1)$$

$$\underline{x}_i = \underline{x}_{i0} + \underline{v}_i t, \quad \underline{v}_i = \underline{v}_{i0} = \text{constant},$$

where \underline{x}_i , \underline{v}_i are the position and velocity of the i^{th} neutron. Various statistical quantities can be calculated using an ensemble function D_N for the initial particle parameters normalized so that

$$\int D_N d\underline{x}_{10} d\underline{v}_{10} \dots d\underline{x}_{N0} d\underline{v}_{N0} = 1, \quad (8.2)$$

and the s -particle distribution is defined by

$$f_s(\underline{x}_{10}, \dots, \underline{v}_{s0}, t) = V^s \int D_N d\underline{x}_{s+1,0} \dots d\underline{v}_{N,0} \quad (8.3)$$

The correlation function of interest to us is that corresponding to fluctuations in the neutron number density, *i.e.*,

$$\begin{aligned}
 \int d\underline{v}_1 d\underline{v}_2 < F(1, t_1) F(2, t_2) > &\equiv \int d\underline{v}_1 d\underline{v}_2 d\underline{x}_{10} \dots dV_{N_0}^{FF} \\
 &= \frac{N}{N_0^2 V} \int dV_{i0} f_1(\underline{v}_{i0}) \delta[\underline{x}_1 - \underline{x}_2 - \underline{v}_{i0}(t_1 - t_2)] \\
 &\quad + \frac{N(N-1)}{N_0^2 V^2} \int d\underline{v}_{i0} d\underline{v}_{j0} f_1(\underline{v}_{i0}) f_1(\underline{v}_{j0}) \\
 &= 1 + \frac{1}{N_0} \frac{1}{(t_1 - t_2)^3} f_1\left(\frac{\underline{x}_1 - \underline{x}_2}{t_1 - t_2}\right) \equiv 1 + \frac{\langle \delta N_0 \delta N_0 \rangle}{N_0^2}
 \end{aligned} \tag{8.4}$$

where N_0 is the average neutron number density and δN_0 the fluctuation in number density.

Assuming a thermal Maxwellian distribution for f_1 (normalized to unity),

$$f_1(\underline{v}) = \frac{1}{(2\pi v_0^2)^{3/2}} \exp\left(-\frac{v^2}{2v_0^2}\right), \tag{8.5}$$

gives

$$\langle \delta N_0(\underline{x}_1, t_1) \delta N_0(\underline{x}_2, t_2) \rangle = \frac{N_0}{(2\pi)^{3/2} [v_0(t_1 - t_2)]^3} \exp\left[-\frac{(\underline{x}_1 - \underline{x}_2)^2}{2v_0^2(t_1 - t_2)^2}\right] \tag{8.6}$$

where v_0 is the neutron thermal velocity and N_0 their average density. For fluctuations at a single point ($x_1 = x_2$) we have,

$$\frac{\langle \delta N_0(1, t_1) \delta N_0(1, t_2) \rangle}{N_0^2} = \frac{1}{N_0 (2\pi)^{3/2} [v_0(t_1 - t_2)]^3} \quad (8.7)$$

We see that this is small except for very fine scale fluctuations with periods of order $1/N_0^{1/3} v_0$ = time it takes a neutron to traverse a mean spacing between neutrons.

Fluctuations in the Number Density of Fission Events and in the Fission Power Density

An approximate estimate of this can be obtained as follows.

Write

$$N_{\text{fiss}} = \text{no. of fissions/cm}^3/\text{sec} = \frac{P_{\text{fiss}}}{Q} \quad (8.8)$$

Thus in a volume ℓ^3 and time τ the average number of fissions is $\ell^3 \tau N_{\text{fiss}}$. Assuming these fission events are uncorrelated, the fluctuation in this number is $(\ell^3 \tau N_{\text{fiss}})^{1/2}$. The corresponding fluctuation in the fission power density can be written

$$\frac{\delta P_{\text{fiss}}}{P_{\text{fiss}}} \approx \frac{1}{\left[\frac{P_{\text{fiss}}}{Q} \ell^3 \tau \right]^{1/2}} \quad (8.9)$$

Now $Q \approx 200 \text{ MeV} \equiv 3.10^{-4} \text{ ergs}$, and if for example we choose $P_{\text{fiss}} \sim 4.10^8 \text{ erg/sec/cm}^3$, it follows that

$$\frac{\delta P_{\text{fiss}}}{P_{\text{fiss}}} \approx \frac{1}{[1.6 \cdot 10^{12} \ell^3 \tau]^{\frac{1}{2}}} \quad (8.10)$$

The amplitude of δP_{fiss} for some length and time scales are shown below:

$$\ell = 10^{-2} \text{ cm}, \tau = 10^{-2} \text{ sec}, \delta P_{\text{fiss}}/P_{\text{fiss}} \approx 10^{-2}$$

$$\ell = 10^{-3} \text{ cm}, \tau = 10^{-1} \text{ sec}, \delta P_{\text{fiss}}/P_{\text{fiss}} \approx 10^{-1}$$

We see that the fission power density has a "grainy" fluctuation level that is appreciable for length scales of $\sim 10^{-2} - 10^{-3} \text{ cm}$, and for the time scales above. For lower power densities this irreducible thermal fluctuation becomes relatively more important. However, since they involve short length scales it is expected that radiation diffusion will considerably decrease the corresponding pressure fluctuations in the plasma.

9. CONCLUDING REMARKS AND RECOMMENDATIONS

The overall conclusions we reached as a result of this analysis can be stated as follows:

- (i) A range of unstable sound waves exists in a fissioning uranium plasma in the wavenumber range $k < k_{\text{crit}}$ (see 4.29). For parameters corresponding to the NASA-Lewis design engine the critical wavelength above which waves are unstable is $\lambda_{\text{crit}} = 2\pi/k_{\text{crit}} \cong 47.5$ cm. Typical e-folding times are .5 sec. The mechanism giving rise to the instability can be described as follows: Consider a standing sound wave in a bounded region of fissioning plasma with a constant background thermal neutron density. In the wave compressions the fission power density, P_{fiss} , increases due to the increased uranium density, and in the rarefactions P_{fiss} decreases. This results in an increased pressure gradient associated with the wave which in turn leads to a transfer of fission power to the wave. It occurs because the wave compression tends to expand more rapidly than it was compressed. However, competing with this is the fact that radiation tends to transport the extra thermal energy out of the wave compressions. Radiation diffusion smooths out the temperature fluctuations of waves more rapidly the shorter their wavelength. This results in a critical wavelength, λ_{crit} , below which waves are stable and above which they are unstable.
- (ii) Waves that are unstable grow in amplitude until limited by nonlinear effects. The mechanism that finally limits their amplitude is the mode coupling of unstable waves in the wavelength range $\lambda > \lambda_{\text{crit}}$ to radiation damped waves in the range $\lambda < \lambda_{\text{crit}}$. Our conclusion is that this mechanism "turns-off" the instability after the wave

amplitudes have reached relatively small values for most systems of interest. Relative pressure fluctuations of order $\delta P/P \sim 10^{-7}$ to 10^{-5} are calculated to occur for a slab of uranium plasma with parameters corresponding to the NASA-Lewis design engine. Consequently this instability is not expected to present a control problem for this engine. It could become a problem, however, if very large devices were designed. We estimate that the pressure fluctuation amplitude would increase as the $5/2$ power of the radius of the uranium plasma core.

- (iii) The dependence of the fission power density on the fluid velocity of the uranium plasma in a cavity in which the neutron gas has a drift motion out of the nozzle can contribute to instability. The mechanism is similar to that for the above instability and is analysed in section 7. It also gives rise to small-amplitude fluctuations in the reactor cavity.

The role of spatial gradients in the neutron density and plasma density has not been included in the above analysis. Since these gradients may also give rise to instability we recommend that these be investigated as a next step in this analysis. Further, the interaction between different flow states of the uranium plasma in the cavity and the fission power should be analyzed (*e.g.*, the hydrogen flow may cause the U plasma core to circulate in convective cells). A computer simulation of these reacting flow problems would be of value.

REFERENCES

There is a voluminous literature on aspects of nuclear plasma reactors and its main application to the gas-cored rocket engine. The first three papers listed contain extensive lists of references.

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